

# KRONECKER QUIVERS, NORMS AND FAMILIES OF CATEGORIES

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**ABSTRACT.** In this paper we introduce several new categorical notions and give many examples.

We start by proving that the moduli space of stability conditions on the derived category of representations of  $K(l)$ , the  $l$ -Kronecker quiver, is biholomorphic to  $\mathbb{C} \times \mathcal{H}$  for  $l \geq 3$ . This produces an example of semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ , where  $\text{Stab}(\mathcal{T})$  is not biholomorphic to  $\text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$  (whereas  $\text{Stab}(\mathcal{T}_1 \oplus \mathcal{T}_2)$  is always biholomorphic to  $\text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$  when  $\text{rank}(K_0(\mathcal{T}_i)) < +\infty$ ).

The above calculations suggest a new notion of a norm. To a triangulated category  $\mathcal{T}$  which has property of a phase gap, we attach a number  $\|\mathcal{T}\|_\varepsilon \in [0, (1 - \varepsilon)\pi]$  depending on a parameter  $\varepsilon \in (0, 1)$ . Under natural assumptions on a semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  holds the inequality  $\|\langle \mathcal{T}_1, \mathcal{T}_2 \rangle\|_\varepsilon \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\}$ .

We compute  $\|\cdot\|_\varepsilon$  in some examples. In particular  $\|D^b(K(l_1))\|_\varepsilon < \|D^b(K(l_2))\|_\varepsilon$  iff  $l_1 < l_2$  and  $3 \leq l_2$ . We believe the norm introduced here is the beginning of a series of new invariants, which deserve deeper studies and understanding. Some of these studies initiate in the last section by relating our norm to the notion of holomorphic family of categories introduced by Kontsevich. We consider family of categories as a non-commutative extension with the norm playing a role similar to the classical notion of degree of an extension in Galois theory. Several questions given by the interplay between towers of sheaves of categories and stability conditions are posed.

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## 1. INTRODUCTION

Motivated by M. Douglas's work in string theory, and especially by the notion of  $\Pi$ -stability, T. Bridgeland defined in [10] a map:

$$(1) \quad \left\{ \begin{array}{c} \text{triangulated} \\ \text{categories} \end{array} \right\} \xrightarrow{\text{Stab}} \left\{ \begin{array}{c} \text{complex} \\ \text{manifolds} \end{array} \right\}.$$

For a triangulated category  $\mathcal{T}$  the associated complex manifold  $\text{Stab}(\mathcal{T})$  is referred to as the space of stability conditions (or the stability space or the moduli space of stability conditions) on  $\mathcal{T}$ .

Bridgeland's manifolds are expected to provide a rigorous understanding of certain moduli spaces arising in string theory. Homological mirror symmetry predicts a parallel between dynamical systems and categories, which is being established in [19], [12], [26], [32], [10], [11], [30], [39]. According to this analogy the stability space plays the role of the Teichmüller space. However while the map (1) being well defined, it is hard to extract global information for the stability spaces. In the present paper we determine explicitly the entire stability space on a new list of examples.

The map (1) behaves well with respect to orthogonal decompositions (see Definition 5.1). This is easy to show and due to lack of an appropriate reference in the literature we have given details on this in Section 5. In particular, there is a bijection

$$(2) \quad \text{Stab}(\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_n) \cong \text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2) \times \cdots \times \text{Stab}(\mathcal{T}_n).$$

which is biholomorphism, when the categories are with finite rank Grothendieck groups. In this paper we show (Theorem 1.1) examples of semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  where  $\text{rank}(K_0(\mathcal{T})) = 2$  however  $\text{Stab}(\mathcal{T})$  is not biholomorphic to  $\text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$ .

The behavior of the map (1) with respect to general semi-orthogonal decomposition (SOD) has been studied in [16]. This study is much more difficult and so far a formula relating  $\text{Stab}(\langle \mathcal{T}_1, \mathcal{T}_2 \rangle)$  and  $\text{Stab}(\mathcal{T}_1)$ ,  $\text{Stab}(\mathcal{T}_2)$  has not been obtained.

In this paper using Bridgeland stability conditions we define (Definition 4.11) for any  $0 < \varepsilon < 1$  a function (the domain is explained below and it does not depend on  $\varepsilon$ ):

$$(3) \quad \left\{ \begin{array}{c} \text{triangulated} \\ \text{categories} \\ \text{with a phase gap} \end{array} \right\} \xrightarrow{\|\cdot\|_\varepsilon} [0, \pi(1 - \varepsilon)]$$

and prove (Theorem 6.1) that if  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  is a semi-orthogonal decomposition in which  $\mathcal{T}$  is proper,<sup>1</sup>  $\text{rank}(K_0(\mathcal{T})) < \infty$ ,  $\mathcal{T}_1$  and  $\mathcal{T}_2$  have phase gaps, then  $\mathcal{T}$  has phase gap as well and

$$(4) \quad \|\langle \mathcal{T}_1, \mathcal{T}_2 \rangle\|_\varepsilon \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\}.$$

For the proof of this inequality we employ the method for gluing of stability conditions in [16], crucial role has also [11, Lemma 4.5] which ensures certain finiteness property of a stability condition with a phase gap.

If  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$  is an orthogonal decomposition with proper  $\mathcal{T}$  and  $\text{rank}(K_0(\mathcal{T})) < \infty$ , then (Corollary 5.6):

$$(5) \quad \|\mathcal{T}_1\|_\varepsilon = 0 \Rightarrow \|\mathcal{T}_1 \oplus \mathcal{T}_2\|_\varepsilon = \|\mathcal{T}_2\|_\varepsilon.$$

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<sup>1</sup>by proper we mean that  $\sum_{i \in \mathbb{Z}} \text{hom}^i(X, Y) < +\infty$  for any two objects  $X, Y$  in  $\mathcal{T}$ .

The function (3) depends on  $\varepsilon \in (0, 1)$ , however the three subsets of its domain determined by the three conditions on the first row in the following table do not depend on  $\varepsilon$  (Lemma 4.16):

Categories with:	$\ \cdot\ _\varepsilon = 0$	$0 < \ \cdot\ _\varepsilon < \pi(1 - \varepsilon)$	$\ \cdot\ _\varepsilon = \pi(1 - \varepsilon)$
examples	for any acyclic quiver $Q$ $D^b(Q)$ is here iff $Q$ is Dynkin or affine	$D^b(K(l_1)) \oplus \cdots \oplus D^b(K(l_N))$ where $N \in \mathbb{Z}_{\geq 1}$ $l_i \geq 3$ for some $i$	$D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ $D^b(\mathbb{P}^n)$ $n \geq 2$ many wild quivers as in Corollary 8.6

The second row contains elements of the corresponding subsets. Further examples can be obtained by using (4) and (5). In particular by blowing up the varieties in the last column one obtains other elements in this column (see Corollary 6.3).

Finally, by relating our norm to the notion of holomorphic family of categories introduced by Kontsevich we suggest a framework in which sequences of holomorphic families of categories are viewed as sequences of extensions of non-commutative manifolds.

1.1. We explain know the new examples where  $\text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$  is not biholomorphic to  $\text{Stab}(\langle \mathcal{T}_1, \mathcal{T}_2 \rangle)$  as well as the examples of categories  $\mathcal{T}$  where we compute (or estimate)  $\|\mathcal{T}\|_\varepsilon$ .

Let us first give some prehistory. By definition each stability condition  $\sigma \in \text{Stab}(\mathcal{T})$  determines a set of non-zero objects in  $\mathcal{T}$  (called *semi-stable objects*) labeled by real numbers (called *phases of the semistable objects*). The semi-stable objects correspond to the so called “BPS” branes in string theory. The set of semi-stable objects will be denoted by  $\sigma^{ss}$ , and  $\phi_\sigma(X) \in \mathbb{R}$  denotes the phase of a semi-stable  $X$ . For any  $\sigma \in \text{Stab}(\mathcal{T})$  we denote by  $P_\sigma^\mathcal{T}$  the subset of the unit circle  $\{\exp(i\pi\phi_\sigma(X)) : X \in \sigma^{ss}\} \subset \mathbb{S}^1$ . A categorical analogue of the density of the set of slopes of closed geodesics on a Riemann surface was proposed in [19]. In [19, section 3] the focus falls on constructing stability conditions for which the set  $P_\sigma$  is dense in a non-trivial arc of the circle. The result is the following characterization of the map (1), when restricted to categories of the form  $D^b(\text{Rep}_k(Q))$  (from now on  $Q$  denotes an acyclic quiver,  $\mathcal{T}$  denotes a triangulated category linear over an algebraically closed field  $k$ ):

(6)	Dynkin quivers (e.g. $\circ \rightarrow \circ$ )	$P_\sigma$ is always finite
	Affine quivers (e.g. $\circ \rightrightarrows \circ$ )	$P_\sigma$ is either finite or has exactly two limit points
	Wild quivers (e.g. $\circ \rightrightarrows \circ$ )	$P_\sigma$ is dense in an arc for a family of stability conditions

In [18, Proposition 3.29] are constructed stability conditions  $\sigma \in \text{Stab}(D^b(Q))$  with two limit points of  $P_\sigma$  for any affine quiver  $Q$  (by  $D^b(Q)$  we mean  $D^b(\text{Rep}_k(Q))$ ).

In [43] and [14] is proved that the stability spaces on Dynkin quivers are contractible, but the affine case is beyond the scope of these papers. For an integer  $l \geq 1$  the  $l$ -Kronicker quiver  $K(l)$  (the quiver with two vertices and  $l$  parallel arrows) is in the first, second, and third row of the table for  $l = 1, 2, 3$ , respectively. In [35] are given arguments that  $\text{Stab}(D^b(K(l)))$  is simply-connected for any  $l \geq 1$ . We develop further in [20], [22] the ideas of Macrì from [35], in particular we give a description of the entire stability space on the acyclic triangular quiver and prove that it is contractible. In [42], [13] and earlier by King is shown that  $\text{Stab}(D^b(K(1)))$  as a complex manifold is  $\mathbb{C}^2$ . Recall that  $D^b(K(2)) \cong D^b(\mathbb{P}^1)$  and  $D^b(K(l))$  for  $l \geq 3$  is equivalent to  $D^b(\mathbb{NP}^{l-1})$ , where  $\mathbb{NP}^l$  is non-commutative projective space, introduced by Kontsevich and Rosenberg in [33] and studied further in [36]. In [40] was shown that  $\text{Stab}(D^b(\mathbb{P}^1)) \cong \mathbb{C}^2$ , and hence  $\text{Stab}(D^b(K(2))) \cong \mathbb{C}^2$

(biholomorphisms). However the question:

$$(7) \quad \text{What is } \text{Stab}(D^b(K(l))) \text{ for } l \geq 3 ?$$

was open after the mentioned papers.

One result of the present paper is:

**Theorem 1.1.** *For each  $l \geq 3$  there exists a biholomorphism  $\text{Stab}(D^b(\text{Rep}_k(K(l)))) \cong \mathbb{C} \times \mathcal{H}$ , where  $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ .*

Thus the map (1) has the same value (up to isomorphism) on all the categories  $\{D^b(K(l))\}_{l \geq 3}$ . Stability conditions on wild quivers whose set of phases are dense in an arc were constructed in [19], however for them the set of phases is still not dense in the entire  $\mathbb{S}^1$ , i.e.  $P_\sigma$  does miss a non-trivial arc, in which case we say for short that  $P_\sigma$  has a gap. In particular all the categories in table (6) are examples of what we call in this paper a *triangulated category with phase gap*, this is a triangulated category  $\mathcal{T}$  for which there exists a full<sup>2</sup>  $\sigma \in \text{Stab}(\mathcal{T})$  whose set of phases  $P_\sigma^\mathcal{T}$  has a gap. Stability conditions whose set of phases is not dense in  $\mathbb{S}^1$  and their relation to so called algebraic stability conditions have been studied in [43]. In particular the results in [43] imply that when  $\text{rank} K_0(\mathcal{T}) < \infty$ , then  $\mathcal{T}$  has a phase gap iff there exists a bounded t-structure in  $\mathcal{T}$  whose heart is of finite length and has finitely many simple objects (Lemma 4.7). Whence the domain of the invariant (3) contains also the CY3 categories discussed in [12].

From the very definition and table (6) one easily derives that for any acyclic quiver  $Q$ :

$$(8) \quad \|D^b(Q)\|_\varepsilon = 0 \iff Q \text{ is Dynkin or affine.}$$

Thus, we can compose the following table, concerning only the quivers  $K(l)$ ,  $l \geq 1$ :

$$(9) \quad \begin{array}{|c|c|c|} \hline Q & \|D^b(Q)\|_\varepsilon & \text{Stab}(D^b(Q)) \\ \hline \circ \rightrightarrows \circ \text{ or } \circ \rightarrow \circ & \|D^b(Q)\|_\varepsilon = 0 & \mathbb{C} \times \mathbb{C} \\ \hline \circ \begin{array}{c} \rightrightarrows \\ \cdot \\ \rightrightarrows \end{array} \circ & \|D^b(Q)\|_\varepsilon > 0 & \mathbb{C} \times \mathcal{H} \\ \hline \end{array}.$$

In the present paper we compute  $\|D^b(K(l))\|_\varepsilon$  for any  $l$  and any  $0 < \varepsilon < 1$ . In particular we derive the following formulas:

$$(10) \quad \|D^b(K(l_1))\|_\varepsilon < \|D^b(K(l_2))\|_\varepsilon \iff l_1 < l_2 \text{ and } 3 \leq l_2$$

$$(11) \quad l \geq 2 \implies \|D^b(K(l))\|_{\frac{1}{2}} = \arccos\left(\frac{2}{l}\right).$$

Combining (8) and table (9) we deduce that for  $l \in \mathbb{N}_{\geq 1}$

$$(12) \quad \|D^b(K(l))\|_\varepsilon = 0 \iff \text{Stab}(D^b(K(l))) \text{ is affine (biholomorphic to } \mathbb{C}^2).$$

We expect that the domains of validity of (4) and (12) can be extended. Regarding (12) we propose:

**Conjecture 1.2.** *Let  $0 < \varepsilon < 1$  and let  $Q$  be any acyclic quiver.*

*The stability space  $\text{Stab}(D^b(Q))$  is affine (of the form  $\mathbb{C}^n$ ) iff  $\|D^b(Q)\|_\varepsilon = 0$ .*

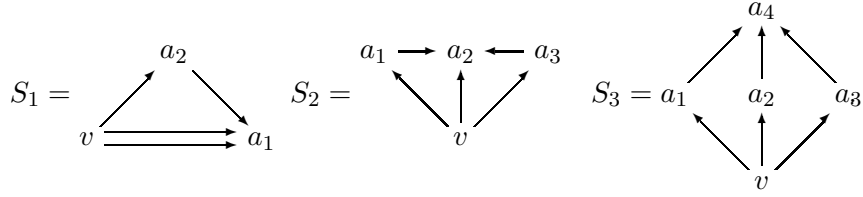
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<sup>2</sup>we recall what is a full stability condition in Section 4.1

We give some criteria ensuring that  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ , they imply that for many of the wild quivers  $Q$  we have  $\|D^b(Q)\|_\varepsilon = \pi(1 - \varepsilon)$  (see Corollary 8.6) and also  $\|D^b(X)\|_\varepsilon = \pi(1 - \varepsilon)$  where  $X$  is  $\mathbb{P}^n$ ,  $n \geq 2$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$  or a smooth algebraic variety obtained from these by blowing up in finitely many points (see Corollary 8.11), for  $n = 1$  we have  $\|D^b(\mathbb{P}^1)\|_\varepsilon = 0$ . Actually, the condition  $\|\mathcal{T}\|_\varepsilon < \pi(1 - \varepsilon)$  imposes restrictions on the full exceptional collections in  $\mathcal{T}$  (see Corollary 8.4).

The criteria for  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$  obtained here do not apply to category of the form  $\mathcal{T} \cong D^b(K(l_1)) \oplus D^b(K(l_2)) \oplus \cdots \oplus D^b(K(l_N))$  and we do prove that  $\|\mathcal{T}\|_\varepsilon < \pi(1 - \varepsilon)$  in this case, which is a generalization of the already discussed wild Kronecker quivers (10).

We expect that the criterion in Corollary 8.3 does not apply to all wild quivers, and we do know that its corollary, Corollary 8.6, cannot be applied to all of them, for example, to the following:



We conjecture, that:

**Conjecture 1.3.** *For  $i = 1, 2, 3$  we have  $0 < \|D^b(S_i)\|_\varepsilon < \pi(1 - \varepsilon)$ .*

1.2. It follows a brief discussion (in this order) on the proof of Theorem 1.1, and of the computations of  $\|\mathcal{T}\|_\varepsilon$ .

For any  $\mathcal{T}$  Bridgeland defined actions of  $\widetilde{GL}^+(2, \mathbb{R})$  (right) and of  $\text{Aut}(\mathcal{T})$  (left) on  $\text{Stab}(\mathcal{T})$ , which commute. The strategy for determining  $\text{Stab}(D^b(\mathbb{P}^1))$  in [40] is to show that the quotient of  $\text{Stab}(D^b(\mathbb{P}^1))$  for an action of  $\mathbb{C} \times \mathbb{Z}$  is isomorphic to  $\mathbb{C}^*$ , where the action of  $\mathbb{C}$  on  $\text{Stab}(D^b(\mathbb{P}^1))$  comes from an embedding of  $\mathbb{C}$  in  $\widetilde{GL}^+(2, \mathbb{R})$  and the action of  $\mathbb{Z}$  comes from the subgroup of  $\text{Aut}(D^b(\mathbb{P}^1))$  generated by the functor  $(\cdot) \otimes \mathcal{O}(1)$ . On the one hand in [40] Okada relies on the commutative geometric nature of  $D^b(K(2)) \cong (D^b(\mathbb{P}^1))$  and on the other hand, implicitly, he relies on the affine nature of the root system of  $K(2)$ , which are obstacles to answer the question (7). In this paper we use the ideas of Okada in [40] and we go through the mentioned obstacles by observing how to apply simple geometry of the action of modular subgroups on  $\mathcal{H}$  and by employing the interplay between exceptional collections and stability conditions developed in [35], [20], [21], [22], [18].

In Section 7.1 we recall facts about the action of the modular group  $\text{SL}(2, \mathbb{Z})$  on  $\mathcal{H}$ , which we need.

In Section 3 we recall what are the actions of  $\mathbb{C}$  and  $\text{Aut}(\mathcal{T})$  on  $\text{Stab}(\mathcal{T})$  for any  $\mathcal{T}$ . It is known that the action of  $\mathbb{C}$  is free and holomorphic ([40], [12]). When  $K_0(\mathcal{T})$  has finite rank, we show that the action of  $\mathbb{C}$  is proper on each connected component of  $\text{Stab}(\mathcal{T})$ , and in particular, when  $\text{Stab}(\mathcal{T})$  is connected, then  $\text{Stab}(\mathcal{T}) \rightarrow \text{Stab}(\mathcal{T})/\mathbb{C}$  is a principal holomorphic  $\mathbb{C}$ -bundle: Proposition 3.2.

Section 7.2 is devoted to the exceptional objects in  $\text{Stab}(\mathcal{T}_l)$ , where  $\mathcal{T}_l = D^b(K(l))$  for  $l \geq 2$ . We utilize here the method of helices [9]. Up to shifts, there is only one helix in  $\mathcal{T}_l$ , which follows from [17]. We denote by  $\{s_i\}_{i \in \mathbb{Z}}$  the helix, for which  $s_1$  is the object in  $\text{Rep}_k(K(l))$ , which is both simple and projective. Lemma 7.4 is the place, where we invoke the action of  $\text{SL}(2, \mathbb{Z})$ , here we give

a formula relating the fractions  $\{\frac{Z(s_{i+1})}{Z(s_i)}\}_{i \in \mathbb{Z}}$  of a central charge<sup>3</sup>  $Z : K_0(\mathcal{T}) \longrightarrow \mathbb{C}$ , it is a simple but important observation for the present paper. For any two exceptional objects  $E_1, E_2$  in  $\mathcal{T}_l$  Lemma 7.5 describes exactly those  $p \in \mathbb{Z}$  for which  $\text{hom}^p(E_1, E_2)$  does not vanish. In [35, Lemma 4.1] is given a statement, but no proof. The statement of Lemma 7.5 is a slight modification of [35, Lemma 4.1] and we give a proof here. By the equivalence  $D^b(\mathbb{P}^1) \cong D^b(K(2))$  the sequence  $\{\mathcal{O}(i)\}_{i \in \mathbb{Z}}$  corresponds, up to translation, to the helix  $\{s_i\}_{i \in \mathbb{Z}}$ . In the non-commutative case  $l \geq 3$  we still have the helix  $\{s_i\}_{i \in \mathbb{Z}}$  and it plays the role of  $\{\mathcal{O}(i)\}_{i \in \mathbb{Z}}$ . Corollary 7.6 is a corollary of [37, Theorem 0.1]) and it ensures existence of a functor  $A_l \in \text{Aut}(\mathcal{T}_l)$  for any  $l \geq 2$ , which is analogous to the functor  $(\cdot) \otimes \mathcal{O}(1)$  for  $l = 2$ , more precisely it satisfies  $A_l(s_i) = s_{i+1}$  for all  $i \in \mathbb{Z}$ . In particular from the subgroup  $\langle A_l \rangle \subset \text{Aut}(\mathcal{T}_l)$  we get an action of  $\mathbb{C} \times \langle A_l \rangle \cong \mathbb{C} \times \mathbb{Z}$  on  $\text{Stab}(\mathcal{T}_l)$  for all  $l \geq 2$ , which for  $l = 2$  coincides with the action on  $\text{Stab}(D^b(\mathbb{P}^1))$  used by Okada for studying  $\text{Stab}(D^b(\mathbb{P}^1))$ .

In sections 7.3 and 7.4, with the help of ideas and results of [35], [20], [21], we go on adapting arguments of Okada about the  $\mathbb{C} \times \mathbb{Z}$ -action on  $\text{Stab}(D^b(\mathbb{P}^1))$  to the non-commutative case, i.e. to the  $\mathbb{C} \times \langle A_l \rangle$ -action on  $\text{Stab}(\mathcal{T}_l)$  for  $l \geq 3$ . We will explain briefly how we utilize the  $\text{SL}(2, \mathbb{Z})$ -action on  $\mathcal{H}$ . [40, p. 497, 498] contain arguments about choosing a representative in the  $\mathbb{C} \times \mathbb{Z}$ -orbit of any stability condition  $\sigma$  for which  $\mathcal{O}, \mathcal{O}(-1)$  are semi-stable and  $0 < \phi_\sigma(\mathcal{O}) < \phi_\sigma(\mathcal{O}(-1)[1]) \leq 1$ . These arguments rely on the fact that for any central charge  $Z$  the vectors  $\{Z(\mathcal{O}(i))\}_{i \in \mathbb{Z}}$  lie on a line in  $\mathbb{C} \cong \mathbb{R}^2$  (see [40, figures 3, 4, 5]), more precisely  $Z(\mathcal{O}(i+1)) - Z(\mathcal{O}(i))$  is the same vector for all  $i \in \mathbb{Z}$ , or in other words  $Z(s_{i+1}) - Z(s_i)$  does not change as  $i$  varies in  $\mathbb{Z}$ , when  $l = 2$ . For  $l \geq 3$  this property of  $\{s_i\}_{i \in \mathbb{Z}}$  fails and the arguments and pictures on [40, p. 497, 498] cannot be applied anymore. We avoid this obstacle by translating this problem to the problem of finding a fundamental domain in  $\mathcal{H}$  of a subgroup of the form  $\langle \alpha_l \rangle \subset \text{SL}(2, \mathbb{Z})$ .<sup>4</sup> This translation is encoded in formula (104) in Lemma 7.13, whose derivation spreads throughout Subsections 7.1, ..., 7.4. The matrix  $\alpha_l$  appears first in Lemma 7.4. For  $l = 2$  this matrix is a parabolic element and for  $l \geq 3$  it is a hyperbolic element in  $\text{SL}(2, \mathbb{Z})$ ,<sup>5</sup> which determines the difference of the type of the fundamental domains of  $\langle \alpha_l \rangle$  (see Figure (2)) in  $\mathcal{H}$ , which in turn determines the difference between the pictures on Figure (3). The colored parts in Figures (3a) and (3b) with only one of the two boundary curves included are in 1-1 correspondence with the set  $\text{Stab}(\mathcal{T}_l)/\mathbb{C} \times \langle A_l \rangle$  for  $l \geq 3$  and  $l = 2$ , respectively. Further properties of the sets given in Figure (3) are derived in Corollary 7.17. In the rest of Section 7.4 is shown how these properties and the presence of the non-trivial real segment  $(-\Delta_l, \Delta_l)$  seen on Figure (3a) imply Theorem 1.1.

In Sections 7.5, 7.6 we compute  $\|\mathcal{T}_l\|_\varepsilon$ . By definition  $\|\mathcal{T}_l\|_\varepsilon$  is the supremum of<sup>6</sup>  $\text{vol}(\overline{P_\sigma^l})/2$  as  $\sigma$  varies in the subset  $\text{Stab}_\varepsilon(\mathcal{T}_l) \subset \text{Stab}(\mathcal{T}_l)$  of those stability conditions  $\sigma$  for which  $P_\sigma^l$  misses at least one closed  $\varepsilon$ -arc (see Definitions 4.3).

Sections 7.2 and 7.3 are a prerequisite for the explicit determining of the set of phases  $P_\sigma^l$  for each  $\sigma \in \text{Stab}(\mathcal{T}_l)$  and each  $l \geq 2$ , which is done in Section 7.5 (Proposition 7.23). It turns out that for  $l \geq 3$  a stability condition has  $\text{vol}(\overline{P_\sigma^l}) \neq 0$  and satisfies  $\sigma \in \text{Stab}_\varepsilon(\mathcal{T}_l)$  iff there exists  $j \in \mathbb{Z}$  such that  $s_j, s_{j+1} \in \sigma^{ss}$  and  $\varepsilon < \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) < 1$ , the set  $P_\sigma^l$  for such a  $\sigma$  is the

<sup>3</sup>i.e. a group homomorphism  $Z : K_0(\mathcal{T}) \longrightarrow \mathbb{C}$  such that  $Z(X) \neq 0$  for any  $X \in \text{Ob}(\mathcal{T})$ .

<sup>4</sup>In our paper fundamental domain is as defined on [38, p. 20], in particular it is a closed subset of  $\mathcal{H}$ .

<sup>5</sup>Which are the hyperbolic and the parabolic elements in  $\text{SL}(2, \mathbb{Z})$  is recalled in Subsection 7.1.1

<sup>6</sup>For a Lebesgue measurable subset  $X \subset \mathbb{S}^1$  we denote by  $\text{vol}(X)$  its Lebesgue measure with  $\text{vol}(\mathbb{S}^1) = 2\pi$ .

set of fractions  $\{n/m : (n, m) \in \Delta_+(K(l))\}$  appropriately embedded in the circle via a function depending on the stability condition. In Lemma 7.22 we shed light on the structure of the set  $\{n/m : (n, m) \in \Delta_+(K(l))\}$  (see formulas (129), (130)) and use it in the proof of Proposition 7.23.

We start Section 7.6 by deriving a formula expressing the non-vanishing numbers  $\text{vol}(\overline{P}_\sigma^l)/2$  as a smooth function depending on  $\frac{|Z(s_{j+1})|}{|Z(s_j)|}$  and  $\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)$  for any  $j \in \mathbb{Z}$  (see Proposition 7.25), which is a straightforward application of the results in Section 7.5. After computing partial derivatives of this function we find that the supremum of  $\text{vol}(\overline{P}_\sigma^l)/2$  as  $\sigma$  varies in  $\text{Stab}_\varepsilon(\mathcal{T}_l)$  is equal to  $\text{vol}(\overline{P}_\sigma^l)/2$  where  $\sigma$  has  $s_j, s_{j+1} \in \sigma^{ss}$ ,  $\frac{|Z(s_{j+1})|}{|Z(s_j)|} = 1$  and  $\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) = \varepsilon$ . The precise formula for  $\|\mathcal{T}_l\|_\varepsilon$  is in Proposition 7.26 and it produces (10), (11). In particular it follows that

$$(13) \quad \lim_{l \rightarrow +\infty} \|\mathcal{T}_l\|_\varepsilon = \pi(1 - \varepsilon).$$

Section 8 contains examples of  $\mathcal{T}$  with  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$  (Corollaries 8.6, 8.11). This section is based on (13) and the observation (Proposition 8.1) that for any exceptional pair  $(E_1, E_2)$  in a proper  $\mathcal{T}$  holds  $\|\langle E_1, E_2 \rangle\|_\varepsilon \geq \|\mathcal{T}_l\|_\varepsilon$  where  $l = \text{hom}^{min}(E_1, E_2)$ . From the arguments leading to these examples it follows that the condition  $\|\mathcal{T}\|_\varepsilon < \pi(1 - \varepsilon)$  imposes restrictions on  $\text{hom}^{min}(E_i, E_j)$  in a full exceptional collection  $(E_0, \dots, E_n)$  (see Corollary 8.4).

Section 9 is devoted to the proof (for any  $N \in \mathbb{Z}_{\geq 1}$  and any  $0 < \varepsilon < 1$ ) of

$$(14) \quad \left\| D^b(K(l_1)) \oplus \dots \oplus D^b(K(l_N)) \right\|_\varepsilon < \pi(1 - \varepsilon).$$

Using the results for the sets  $P_\sigma^l$  from Sections 7.5, 7.6 we show here that, whenever  $P_\sigma^l$  is contained in  $C \cup -C$  for an open arc  $C \subset \mathbb{S}^1$  with length less than  $\pi$ , then for some closed arc  $p_\sigma^l \subset C \cap \overline{P}_\sigma^l$  the set  $\overline{P}_\sigma^l \setminus (p_\sigma^l \cup -p_\sigma^l)$  is at most countable, and furthermore, provided that the length of  $C$  is fixed, we show that when some of the end points of  $p_\sigma^l$  is very close to some of the end points of  $C$ , then  $p_\sigma^l$  itself has very small length (Corollary 9.3). Due to the fact, proven in Section 5, that for any orthogonal decomposition  $\mathcal{T} = \mathcal{T}_1 \oplus \dots \oplus \mathcal{T}_n$  and any  $\sigma \in \text{Stab}(\mathcal{T})$  holds  $P_\sigma^T = \bigcup_{i=1}^n P_{\sigma_i}^{T_i}$ , where  $(\sigma_1, \dots, \sigma_n)$  is the value of the map (2) at  $\sigma$  (see Proposition 5.2 and Corollary 5.5), the proof of (14) reduces to proving that the measure of union of arcs  $\bigcup_{i=1}^n p_\sigma^{l_i} \subset C$  of the type explained above, cannot become arbitrary close to the length of  $C$ . Having proved this for one arc (in Section 7.6) we perform induction and the tool for the induction step is the already discussed Corollary 9.3.

In section 10 we initiate some studies by relating our norm to the notion of holomorphic family of categories introduced by Kontsevich. Several questions and conjectures are posed.

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## 2. NOTATIONS

In this paper the letters  $\mathcal{T}$  and  $\mathcal{A}$  denote a triangulated category and an abelian category, respectively, linear over an algebraically closed field  $k$ . The shift functor in  $\mathcal{T}$  is designated by  $[1]$ . We write  $\text{Hom}^i(X, Y)$  for  $\text{Hom}(X, Y[i])$  and  $\text{hom}^i(X, Y)$  for  $\dim_k(\text{Hom}(X, Y[i]))$ , where  $X, Y \in \mathcal{T}$ . For  $X, Y \in \mathcal{A}$ , writing  $\text{Hom}^i(X, Y)$ , we consider  $X, Y$  as elements in  $\mathcal{T} = D^b(\mathcal{A})$ , i.e.  $\text{Hom}^i(X, Y) = \text{Ext}^i(X, Y)$ .

A triangulated category  $\mathcal{T}$  is called *proper* if  $\sum_{i \in \mathbb{Z}} \text{hom}^i(X, Y) < +\infty$  for any two objects  $X, Y$  in  $\mathcal{T}$ . For  $X, Y \in \mathcal{T}$  in a proper  $\mathcal{T}$ , we denote:

$$(15) \quad \text{hom}^{\min}(X, Y) = \begin{cases} \text{hom}^i(X, Y) & \text{if } i = \min\{j : \text{hom}^j(X, Y) \neq 0\} > -\infty \\ 0 & \text{otherwise.} \end{cases}$$

We write  $\langle S \rangle \subset \mathcal{T}$  for the triangulated subcategory of  $\mathcal{T}$  generated by  $S$ , when  $S \subset \text{Ob}(\mathcal{T})$ .

An *exceptional object* is an object  $E \in \mathcal{T}$  satisfying  $\text{Hom}^i(E, E) = 0$  for  $i \neq 0$  and  $\text{Hom}(E, E) = k$ . We denote by  $\mathcal{A}_{exc}$ , resp.  $D^b(\mathcal{A})_{exc}$ , the set of all exceptional objects of  $\mathcal{A}$ , resp. of  $D^b(\mathcal{A})$ .

An *exceptional collection* is a sequence  $\mathcal{E} = (E_0, E_1, \dots, E_n) \subset \mathcal{T}_{exc}$  satisfying  $\text{hom}^*(E_i, E_j) = 0$  for  $i > j$ . If in addition we have  $\langle \mathcal{E} \rangle = \mathcal{T}$ , then  $\mathcal{E}$  will be called a full exceptional collection. For a vector  $\mathbf{p} = (p_0, p_1, \dots, p_n) \in \mathbb{Z}^{n+1}$  we denote  $\mathcal{E}[\mathbf{p}] = (E_0[p_0], E_1[p_1], \dots, E_n[p_n])$ . Obviously  $\mathcal{E}[\mathbf{p}]$  is also an exceptional collection. The exceptional collections of the form  $\{\mathcal{E}[\mathbf{p}] : \mathbf{p} \in \mathbb{Z}^{n+1}\}$  will be said to be shifts of  $\mathcal{E}$ .

If an exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n) \subset \mathcal{T}_{exc}$  satisfies  $\text{hom}^k(E_i, E_j) = 0$  for any  $i, j$  and for  $k \neq 0$ , then it is said to be *strong exceptional collection*.

For two exceptional collections  $\mathcal{E}_1, \mathcal{E}_2$  of equal length we write  $\mathcal{E}_1 \sim \mathcal{E}_2$  if  $\mathcal{E}_2 \cong \mathcal{E}_1[\mathbf{p}]$  for some  $\mathbf{p} \in \mathbb{Z}^{n+1}$ .

An abelian category  $\mathcal{A}$  is said to be *hereditary*, if  $\text{Ext}^i(X, Y) = 0$  for any  $X, Y \in \mathcal{A}$  and  $i \geq 2$ , it is said to be of *finite length*, if it is Artinian and Noetherian.

By  $Q$  we denote an acyclic quiver and by  $D^b(\text{Rep}_k(Q))$ , or just  $D^b(Q)$ , - the derived category of the category of representations of  $Q$ .

For an integer  $l \geq 1$  the  $l$ -Kronicker quiver (the quiver with two vertices and  $l$  parallel arrows) will be denoted by  $K(l)$ .

The real and the imaginary parts of a complex number  $z \in \mathbb{C}$  will be denoted by  $\Re(z)$  and  $\Im(z)$ , respectively, and by  $\mathcal{H}$  we denote the upper half plane, i. e.  $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ . For any complex number  $z \in \mathcal{H}$  we denote by  $\text{Arg}(z)$  the unique  $\phi \in (0, \pi)$  satisfying  $z = |z| \exp(i\phi)$ .

For a complex number  $z = (a + ib)$ ,  $a, b \in \mathbb{R}$  we denote  $\Im(z) = b$ ,  $\Re(z) = a$ .

The letter  $\mathbb{H}$  will denote the upper half plane with the negative real axis included, i. e.  $\mathbb{H} = \{r \exp(i\pi t) : r > 0 \text{ and } 0 < t \leq 1\}$ .

For an element  $\alpha$  in a group  $G$  we denote by  $\langle \alpha \rangle \subset G$  the subgroup  $\langle \alpha \rangle = \{\alpha^i\}_{i \in \mathbb{Z}}$ .

If  $A \subset B$  are subsets in a top. space  $X$ , we denote by  $\text{Bd}_B(A)$  the boundary of  $A$  w.r. to  $B$ , and by  $L_B(A)$  the set of limit points of  $A$  w.r. to  $B$ .

## 3. ON BRIDGELAND STABILITY CONDITIONS

We use freely the axioms and notations on stability conditions introduced by Bridgeland in [10] and some additional notations used in [20, Subsection 3.2]. In particular, the underlying set of the manifold  $\text{Stab}(\mathcal{T})$  is the set of locally finite stability conditions on  $\mathcal{T}$  and for  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$



we denote by  $\sigma^{ss}$  the set of  $\sigma$ -semistable objects, i. e.

$$(16) \quad \sigma^{ss} = \cup_{t \in \mathbb{R}} \mathcal{P}(t) \setminus \{0\}.$$

Also for a heart  $\mathcal{A}$  of bounded t-structure in  $\mathcal{T}$  we denote by  $\mathbb{H}^{\mathcal{A}} \subset \text{Stab}(\mathcal{T})$  the subset of the stability conditions  $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$  for which  $\mathcal{P}(0, 1] = \mathcal{A}$  (see [18, Definition 2.28]).

Recall that one of Bridgeland's axioms [10] is: for any nonzero  $X \in \text{Ob}(\mathcal{T})$  there exists a diagram of distinguished triangles called *Harder-Narasimhan filtration*:

$$(17) \quad \begin{array}{ccccccc} 0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \rightarrow \dots \rightarrow & E_{n-1} & \xrightarrow{\quad} & E_n = X \\ & \searrow \text{dashed} & \swarrow & & \swarrow & & \searrow \text{dashed} & & \swarrow \\ & & A_1 & & A_2 & & & & A_n \end{array}$$

where  $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$ ,  $t_1 > t_2 > \dots > t_n$  and  $A_i$  is non-zero object for any  $i = 1, \dots, n$  (the non-vanishing condition makes the factors  $\{A_i \in \mathcal{P}(t_i)\}_{i=1}^n$  unique up to isomorphism). Following [10] we denote  $\phi_{\sigma}^{-}(X) := t_n$ ,  $\phi_{\sigma}^{+}(X) := t_1$ , and the phase of a semistable object  $A \in \mathcal{P}(t) \setminus \{0\}$  is denoted by  $\phi_{\sigma}(A) := t$ . The positive integer:

$$(18) \quad m_{\sigma}(X) = \sum_{i=1}^n |Z(A_i)|$$

is called the mass of  $X$  w.r. to  $\sigma$  ([10, p.332]). We will use also the following axioms [10]:

$$(19) \quad X \in \sigma^{ss} \quad \Rightarrow \quad Z(X) = m_{\sigma}(X) \exp(i\pi\phi_{\sigma}(X)), \quad m_{\sigma}(X) = |Z(X)| > 0$$

$$(20) \quad X, Y \in \sigma^{ss} \quad \phi_{\sigma}(X) > \phi_{\sigma}(Y) \quad \Rightarrow \quad \text{Hom}(X, Y) = 0.$$

Finally we note that:

$$(21) \quad \begin{aligned} X \cong X_1 \oplus X_2 &\Rightarrow \begin{aligned} m_{\sigma}(X) &= m_{\sigma}(X_1) + m_{\sigma}(X_2) \\ \phi_{\sigma}^{-}(X) &= \min\{\phi_{\sigma}^{-}(X_1), \phi_{\sigma}^{-}(X_2)\} \\ \phi_{\sigma}^{+}(X) &= \max\{\phi_{\sigma}^{+}(X_1), \phi_{\sigma}^{+}(X_2)\} \end{aligned} \end{aligned}$$

which follows easily from (18) and the arguments for the proof of [18, Lemma 2.25].

### 3.1. Actions on $\text{Stab}(\mathcal{T})$ .

**3.1.1. The universal covering group of  $GL^+(2, \mathbb{R})$ .** The universal covering group  $\widetilde{GL}^+(2, \mathbb{R})$  of  $GL^+(2, \mathbb{R})$  can be constructed as follows (we point the steps without proving them). First step is to show that the following set with the specified bellow operations and metric is a topological group:

$$(22) \quad \widetilde{GL}^+(2, \mathbb{R}) = \left\{ (G, \psi) : \begin{aligned} &G \in GL^+(2, \mathbb{R}), \quad \psi \in C^{\infty}(\mathbb{R}) \\ &\forall t \in \mathbb{R} \quad \psi'(t) > 0, \quad \psi(t+1) = \psi(t) + 1, \quad \frac{G(\exp(i\pi t))}{|G(\exp(i\pi t))|} = \exp(i\pi\psi(t)) \end{aligned} \right\}$$

$$(23) \quad \text{unit element: } (\text{Id}_{\mathbb{C}}, \text{Id}_{\mathbb{R}})$$

$$(24) \quad \text{multiplication: } ((G_1, \psi_1), (G_2, \psi_2)) \mapsto (G_1 \circ G_2, \psi_1 \circ \psi_2)$$

$$(25) \quad \text{inverse element: } (G, \psi) \mapsto (G^{-1}, \psi^{-1})$$

$$(26) \quad \text{metric: } d((G_1, \psi_1), (G_2, \psi_2)) = \sup_{t \in \mathbb{R}} \{|G_1(\exp(i\pi t)) - G_2(\exp(i\pi t))|, |\psi_1(t) - \psi_2(t)|\}.$$

Second step is to show that the following is a covering map:

$$(27) \quad \widetilde{GL}^+(2, \mathbb{R}) \xrightarrow{\pi} GL^+(2, \mathbb{R}) \quad (G, \psi) \mapsto G.$$

The subset  $U_\varepsilon = \{G \in GL^+(2, \mathbb{R}); \sup_{t \in \mathbb{R}} \{|G(\exp(i\pi t)) - \exp(i\pi t)|\} < \sin(\pi\varepsilon)\}^7$  is evenly covered by a family of open subsets  $\{(G, \psi); G \in U_\varepsilon \quad \sup_{t \in \mathbb{R}} |\psi(t) - t - 2k| < \varepsilon\}$  indexed by  $k \in \mathbb{Z}$  for small enough  $\varepsilon$ . In particular one obtains a structure or a Lie Group on  $\widetilde{GL}^+(2, \mathbb{R})$  such that  $\pi$  is a morphism of Lie groups.

Finally, one can show that  $\widetilde{GL}^+(2, \mathbb{R})$  is simply connected by recalling that  $\pi_1(GL^+(2, \mathbb{R})) \cong \mathbb{Z}$  is generated by  $\mathbb{S}^1 = SO(2) \subset GL^+(2, \mathbb{R})$  and then by finding the lifts of this path in  $\widetilde{GL}^+(2, \mathbb{R})$ .

**Remark 3.1.** *For any  $0 < \varepsilon < 1$ ,  $0 < \varepsilon' < 1$  there exists unique  $g_{\varepsilon, \varepsilon'} = (G, \psi) \in \widetilde{GL}^+(2, \mathbb{R})$  such that  $G^{-1}(1) = 1$  and  $G^{-1}(\exp(i\pi\varepsilon)) = \exp(i\pi\varepsilon')$  and  $\psi(0) = 0, \psi(1) = 1, \psi(\varepsilon') = \varepsilon$ , in particular :*

$$(28) \quad \psi([0, \varepsilon']) = [0, \varepsilon], \quad \psi([\varepsilon', 1]) = [\varepsilon, 1]$$

Furthermore,  $(g_{\varepsilon, \varepsilon'})^{-1} = g_{\varepsilon', \varepsilon}$ .

The right action of  $\widetilde{GL}^+(2, \mathbb{R})$  on  $\text{Stab}(\mathcal{T})$  is defined by (recall [10]):

$$(29) \quad \text{Stab}(\mathcal{T}) \times \widetilde{GL}^+(2, \mathbb{R}) \rightarrow \text{Stab}(\mathcal{T}) \quad ((Z, \mathcal{P}), (G, \psi)) \mapsto (Z, \mathcal{P}) \cdot (G, \psi) = (G^{-1} \circ Z, \mathcal{P} \circ \psi).$$

Using the formula (26) determining the topology on  $\widetilde{GL}^+(2, \mathbb{R})$  and the basis of the topology in  $\text{Stab}(\mathcal{T})$  explained on [10, p. 335] one can show that the function in (29) is continuous.

We recall also (see [10, Theorem 1.2]) that the projection  $\text{Stab}(\mathcal{T}) \xrightarrow{\text{proj}} \text{Hom}(K_0(\mathcal{T}), \mathbb{C})$ ,  $\text{proj}(Z, \mathcal{P}) = Z$  restricts to a local biholomorphism between each connected component of  $\text{Stab}(\mathcal{T})$  and a corresponding vector subspace of  $\text{Hom}(K_0(\mathcal{T}), \mathbb{C})$  with a well defined linear topology (when  $\text{rank}(K_0(\mathcal{T})) < +\infty$  this is the ordinary linear topology). Note also that the results in [10] imply that  $\text{Stab}(\mathcal{T})$  is locally path connected (follows from the results in [10, Section 6] and [10, Theorem 7.1]), therefore the components and the path components of  $\text{Stab}(\mathcal{T})$  coincide and they are open subsets in  $\text{Stab}(\mathcal{T})$ .

Finally, assume for simplicity that  $\text{rank}(K_0(\mathcal{T})) < +\infty$ . Due to continuity of (29) it follows that for each connected component  $\Sigma$  of  $\text{Stab}(\mathcal{T})$  the action (29) restricts to a continuous action  $\Sigma \times \widetilde{GL}^+(2, \mathbb{R}) \rightarrow \Sigma$  and it is easy to show that there is a commutative diagram:

$$(30) \quad \begin{array}{ccc} \Sigma \times \widetilde{GL}^+(2, \mathbb{R}) & \longrightarrow & \Sigma \\ \text{proj}_1 \times \pi \downarrow & & \text{proj}_1 \downarrow \\ V(\Sigma) \times GL^+(2, \mathbb{R}) & \longrightarrow & V(\Sigma) \end{array}$$

where  $V(\Sigma) \subset \text{Hom}(K_0(\mathcal{T}), \mathbb{C})$  is the corresponding to  $\Sigma$  vector subspace, such that the vertical arrows are local diffeomorphisms (the right arrow is local biholomorphism), and the lower horizontal arrow is an action of the form  $(A, G) \mapsto A \circ G^{-1}$  on  $V(\Sigma)$ . Now it follows that the upper horizontal arrow is smooth, and therefore (29) is smooth as well.

**3.1.2. The action of  $\mathbb{C}$ .** There is a Lie group homomorphism  $\mathbb{C} \rightarrow \widetilde{GL}^+(2, \mathbb{R})$  given by  $\lambda \mapsto (e^{-\lambda}, \text{Id}_{\mathbb{R}} - \Im(\lambda)/\pi)$ . And composing the action (29) with this homomorphism results in the action (31) below. This action is free [40, Definition 2.3, Proposition 4.1]. It is easy to show that for any

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<sup>7</sup> neighborhood of  $\text{Id}_{\mathbb{C}} \in GL^+(2, \mathbb{R})$

$X \in \mathcal{T}$ ,  $\sigma \in \text{Stab}(\mathcal{T})$ ,  $z \in \mathbb{C}$  hold the properties in (32), (33) below, and the HN filtrations of  $X$  w.r. to  $\sigma$  and to  $z \star \sigma$  are the same:

$$(31) \quad \mathbb{C} \times \text{Stab}(\mathcal{T}) \xrightarrow{\star} \text{Stab}(\mathcal{T}) \quad z \star (Z, \{\mathcal{P}(t)\}_{t \in \mathbb{R}}) = (e^z Z, \{\mathcal{P}(t - \Im(z)/\pi)\}_{t \in \mathbb{R}})$$

$$(32) \quad (z \star \sigma)^{ss} = \sigma^{ss} \quad \phi_{z \star \sigma}(X) = \phi_\sigma(X) + \Im(z)/\pi \quad X \in \sigma^{ss}$$

$$(33) \quad \phi_{z \star \sigma}^\pm(X) = \phi_\sigma^\pm(X) + \Im(z)/\pi; \quad m_{z \star \sigma}(X) = e^{\Re(z)} m_\sigma(X).$$

**Proposition 3.2.** *Let  $K_0(\mathcal{T})$  have finite rank and let  $\Sigma \subset \text{Stab}(\mathcal{T})$  be a connected component. Then the action (31) restricted to  $\Sigma$  is proper. In particular,  $\Sigma/\mathbb{C}$  with the quotient topology carries a structure of a complex manifold, s. t. the projection  $\text{pr} : \Sigma \rightarrow \Sigma/\mathbb{C}$  is a holomorphic  $\mathbb{C}$ -principal bundle of complex dimension  $\dim_{\mathbb{C}}(\text{Stab}(\mathcal{T})) - 1$ .*

*Proof.* The action is holomorphic (now we get a diagram (30) with  $\mathbb{C}$  instead of  $\widetilde{GL}^+(2, \mathbb{R})$ ,  $\mathbb{C}^\star$  instead of  $GL^+(2, \mathbb{R})$ , and now both the vertical arrows are local biholomorphisms, whereas the lower horizontal arrow is holomorphic) and free (see e.g. [40, Proposition 4.1]). If we show that the function

$$(34) \quad \gamma : \mathbb{C} \times \Sigma \rightarrow \Sigma \times \Sigma \quad (\lambda, \sigma) \mapsto (\lambda \star \sigma, \sigma)$$

is proper, then the proposition follows from [31, Proposition 1.2]. Let  $K_i \subset \Sigma$ ,  $i = 1, 2$  be two compact subsets. Since  $\text{Stab}(\mathcal{T}) \times \text{Stab}(\mathcal{T})$  is locally compact, it is enough to show that  $\gamma^{-1}(K_1 \times K_2) \subset \mathbb{C} \times \Sigma$  is compact.

[10, Proposition 8.1.] says that assigning to any two  $\sigma_1, \sigma_2 \in \text{Stab}(\mathcal{T})$  the following:

$$(35) \quad d(\sigma_1, \sigma_2) = \sup_{0 \neq X \in \mathcal{T}} \left\{ |\phi_{\sigma_1}^-(X) - \phi_{\sigma_2}^-(X)|, |\phi_{\sigma_1}^+(X) - \phi_{\sigma_2}^+(X)|, \left| \log \frac{m_{\sigma_2}(X)}{m_{\sigma_1}(X)} \right| \right\} \in [0, +\infty]$$

defines a generalized metric whose topology coincides with the topology of  $\text{Stab}(\mathcal{T})$ . Furthermore, since  $\Sigma$  is a connected component, it follows that  $d(\sigma_1, \sigma_2) < +\infty$  for any two  $\sigma_1, \sigma_2 \in \Sigma$ , hence  $d$  is a usual metric on  $\Sigma$ . In particular, the function  $d : \Sigma \times \Sigma \rightarrow \mathbb{R}$  is continuous, hence there exists  $M \in \mathbb{R}_{>0}$ , s.t.

$$(36) \quad \forall x, y \in K_1 \times K_2 \quad d(x, y) \leq M.$$

From (33) and (35) we see that

$$(37) \quad \forall \sigma \in \text{Stab}(\mathcal{T}) \quad \forall \lambda \in \mathbb{C} \quad \max \{ |\Re(\lambda)|, |\Im(\lambda)/\pi| \} \leq d(\sigma, \lambda \star \sigma).$$

Assume that  $(\lambda, \sigma) \in \gamma^{-1}(K_1 \times K_2)$ , then  $\gamma(\lambda, \sigma) = (\lambda \star \sigma, \sigma) \in K_1 \times K_2$ , therefore  $\sigma \in K_2$  and by (36), (37) it follows that  $\max \{ |\Re(\lambda)|, |\Im(\lambda)/\pi| \} \leq d(\lambda \star \sigma, \sigma) \leq M$ , thus we obtain:

$$(38) \quad \gamma^{-1}(K_1 \times K_2) \subset \{ \lambda \in \mathbb{C}; |\Re(\lambda)| \leq M, |\Im(\lambda)| \leq \pi M \} \times K_2,$$

therefore  $\gamma^{-1}(K_1 \times K_2)$  is a closed subset of a compact subset, hence  $\gamma^{-1}(K_1 \times K_2)$  is compact and the proposition is proved.  $\square$

**Corollary 3.3.** *Let  $\mathcal{T}$  has a contractible stability space, and let  $\dim_{\mathbb{C}}(\text{Stab}(\mathcal{T})) = \text{rank}(K_0(\mathcal{T})) = 2$ . Then  $\text{Stab}(\mathcal{T})$  is biholomorphic to one of the two:  $\mathbb{C} \times \mathbb{C}$  or  $\mathbb{C} \times \mathcal{H}$  and the quotient map  $\text{pr} : \text{Stab}(\mathcal{T}) \mapsto \text{Stab}(\mathcal{T})/\mathbb{C}$  is a trivial  $\mathbb{C}$ -principal bundle.*

*Proof.* Proposition 3.2 imply that  $\mathbf{pr} : \text{Stab}(\mathcal{T}) \mapsto \text{Stab}(\mathcal{T})/\mathbb{C}$  is a  $\mathbb{C}$ -principal bundle and now we are given that the total space is contractible and the base is a one dimensional connected complex manifold. Now from the long sequence of the homotopy groups associated to a fibration one can deduce that  $\text{Stab}(\mathcal{T})/\mathbb{C}$  is contractible one dimensional complex manifold (see e.g. [48, p. 82] for more details on the arguments). By uniformisation theorem  $\text{Stab}(\mathcal{T})/\mathbb{C}$  is biholomorphic either to  $\mathbb{C}$  or to  $\mathcal{H}$ . Now the corollary follows from the theorem that every fiber bundle over a noncompact Riemann surface is trivial, provided the structure group  $G$  is connected. ([29], [44]).  $\square$

3.1.3. *The action of  $\text{Aut}(\mathcal{T})$ .* There is a left action of the group of exact autoequivalences  $\text{Aut}(\mathcal{T})$  on  $\text{Stab}(\mathcal{T})$ , which commutes with the action (31) [10, Lemma 8.2]. This action is determined as follows:

$$(39) \quad \text{Aut}(\mathcal{T}) \times \text{Stab}(\mathcal{T}) \ni (\Phi, (Z, \{\mathcal{P}(t)\}_{t \in \mathbb{R}})) \mapsto \left( Z \circ [\Phi]^{-1}, \left\{ \overline{\Phi(\mathcal{P}(t))} \right\}_{t \in \mathbb{R}} \right) \in \text{Stab}(\mathcal{T}),$$

where  $[\Phi] : K_0(\mathcal{T}) \rightarrow K_0(\mathcal{T})$  is the induced isomorphism (we will often omit specifying the square brackets) and  $\overline{\Phi(\mathcal{P}(t))}$  is the full isomorphism closed subcategory containing  $\Phi(\mathcal{P}(t))$ .

When  $\text{rank}(K_0(\mathcal{T})) < +\infty$ , it is easy to show that  $\text{Aut}(\mathcal{T})$  acts via (39) biholomorphically on  $\text{Stab}(\mathcal{T})$ .

For any  $\Phi \in \text{Aut}(\mathcal{T})$ ,  $\sigma \in \text{Stab}(\mathcal{T})$  let us denote  $\sigma = (\mathcal{P}_\sigma, Z_\sigma)$ ,  $\Phi \cdot \sigma = (\mathcal{P}_{\Phi \cdot \sigma}, Z_{\Phi \cdot \sigma})$ , then we have:

$$(40) \quad (\Phi \cdot \sigma)^{ss} = \overline{\Phi(\sigma^{ss})} \quad \phi_{\Phi \cdot \sigma}(X) = \phi_\sigma(\Phi^{-1}X) \quad X \in (\Phi \cdot \sigma)^{ss}$$

$$(41) \quad Z_{\Phi \cdot \sigma}(X) = Z_\sigma(\Phi^{-1}(X)) \quad X \in \mathcal{T}.$$

#### 4. TRIANGULATED CATEGORIES WITH PHASE GAPS AND THEIR NORMS

4.1. **Full stability conditions.** We start this section by recalling what is meant when saying that a stability condition is full.

*Full stability condition* on  $K3$  surface is defined in [10, Definition 4.2]. Analogous definition can be given for any triangulated category  $\mathcal{T}$  and locally finite stability condition whose central charge factors through a given group homomorphism  $ch : K_0(\mathcal{T}) \rightarrow \mathbb{Z}^n$ .

When  $K_0(\mathcal{T})$  has finite rank, we choose always the trivial homomorphism  $K_0(\mathcal{T}) \rightarrow K_0(\mathcal{T})$ . Now the projection  $\text{Stab}(\mathcal{T}) \xrightarrow{\text{proj}} \text{Hom}(K_0(\mathcal{T}), \mathbb{C})$ ,  $\text{proj}(Z, \mathcal{P}) = Z$  restricts to a local biholomorphism between each connected component of  $\text{Stab}(\mathcal{T})$  and a corresponding vector subspace of  $\text{Hom}(K_0(\mathcal{T}), \mathbb{C})$  (see [10, Theorem 1.2]). A stability condition  $\sigma \in \text{Stab}(\mathcal{T})$  in this case is a full stability condition, if the vector subspace of  $\text{Hom}(K_0(\mathcal{T}), \mathbb{C})$  corresponding to the connected component  $\Sigma$  containing  $\sigma$  is the entire  $\text{Hom}(K_0(\mathcal{T}), \mathbb{C})$ , which is equivalent to the equality  $\dim_{\mathbb{C}}(\Sigma) = \text{rank}(K_0(\mathcal{T}))$ .

As we will see later all stability conditions on  $K(l)$  are full, for all  $l \geq 1$  (see table 9). It is reasonable to hope that, whenever  $\text{Stab}(\mathcal{T}) \neq \emptyset$ , there are always full stability conditions on  $\mathcal{T}$  and, to the best of our knowledge, there are no counterexamples of this statement so far.

4.2. **The  $\varepsilon$ -norm of a triangulated category.** Recall that for  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$  we denote (see [19, Section 3]):

$$(42) \quad P_\sigma^\mathcal{T} = \{\exp(i\pi\phi_\sigma(X)) : X \in \sigma^{ss}\} = \{\exp(i\pi t) : t \in \mathbb{R} \text{ and } \mathcal{P}(t) \neq \{0\}\}.$$

Here we will use also the notation:

$$(43) \quad \tilde{P}_\sigma^\mathcal{T} = \{t \in \mathbb{R} : \mathcal{P}(t) \neq \{0\}\} \Rightarrow P_\sigma^\mathcal{T} = \exp\left(i\pi\tilde{P}_\sigma^\mathcal{T}\right).$$

The sets  $P_\sigma^\mathcal{T}$  and  $\tilde{P}_\sigma^\mathcal{T}$  satisfy  $P_\sigma^\mathcal{T} = -P_\sigma^\mathcal{T}$ ,  $\tilde{P}_\sigma^\mathcal{T} + 1 = \tilde{P}_\sigma^\mathcal{T}$ . In particular the closures  $\overline{P_\sigma^\mathcal{T}}$ ,  $\overline{\tilde{P}_\sigma^\mathcal{T}}$  satisfy:

$$(44) \quad \text{vol}(\overline{P_\sigma^\mathcal{T}}) = 2\pi\mu(\overline{\tilde{P}_\sigma^\mathcal{T}} \cap [0, 1]) = 2\pi \int_{[0,1] \cap \overline{\tilde{P}_\sigma^\mathcal{T}}} d\mu,$$

where  $\mu$  is the Lebesgue measure in  $\mathbb{R}$  and  $\text{vol}$  is the corresponding measure in  $\mathbb{S}^1$  with  $\text{vol}(\mathbb{S}^1) = 2\pi$ . Due to (29), (32), for any  $z \in \mathbb{C}$ , any  $g = (G, \psi) \in \widetilde{GL}^+(2, \mathbb{R})$ , and any  $\sigma \in \text{Stab}(\mathcal{T})$  we have:

$$(45) \quad P_{(z \star \sigma)}^\mathcal{T} = \exp(i\Im(z))P_\sigma^\mathcal{T} \quad \tilde{P}_{(\sigma \cdot g)}^\mathcal{T} = \psi^{-1}(\tilde{P}_\sigma^\mathcal{T}).$$

**Definition 4.1.** Let  $0 < \varepsilon < 1$ . Any subset of  $\mathbb{S}^1$  of the form  $\exp(i\pi[a, a + \varepsilon])$ , where  $a \in \mathbb{R}$  will be referred to as a **closed  $\varepsilon$ -arc** in  $\mathbb{S}^1$ .

**Remark 4.2.** The action of  $\text{Aut}(\mathcal{T})$  on  $\text{Stab}(\mathcal{T})$  was recalled in the end of the previous section. Following this definition one defines straightforwardly a biholomorphism  $[F] : \text{Stab}(\mathcal{T}_1) \rightarrow \text{Stab}(\mathcal{T}_2)$  for any equivalence  $F$  between triangulated categories  $\mathcal{T}_1$  and  $\mathcal{T}_2$  satisfying  $P_{[F](\sigma)}^{\mathcal{T}_2} = P_\sigma^{\mathcal{T}_1}$  for each  $\sigma \in \text{Stab}(\mathcal{T}_1)$ .

In Definition 4.11 we will use the following subset of the set of stability conditons:

**Definition 4.3.** For any  $0 < \varepsilon < 1$  and any triangulated category  $\mathcal{T}$  we denote:

$$\text{Stab}_\varepsilon(\mathcal{T}) = \{\sigma \in \text{Stab}(\mathcal{T}) : \sigma \text{ is full and } \mathbb{S}^1 \setminus P_\sigma^\mathcal{T} \text{ contains a closed } \varepsilon \text{ arc}\}$$

$$\text{Stab}_{[a, a+\varepsilon]}(\mathcal{T}) = \{\sigma \in \text{Stab}(\mathcal{T}) : \sigma \text{ is full and } \tilde{P}_\sigma^\mathcal{T} \cap [a, a + \varepsilon] = \emptyset\}.$$

It is obvious that (recall also (45)):

$$(46) \quad \text{Stab}_\varepsilon(\mathcal{T}) = \cup_{a \in \mathbb{R}} \text{Stab}_{[a, a+\varepsilon]}(\mathcal{T}) = \mathbb{C} \star \text{Stab}_{[0, \varepsilon]}(\mathcal{T})$$

The next simple observation is:

**Lemma 4.4.** Let  $g_{\varepsilon, \varepsilon'} \in \widetilde{GL}^+(2, \mathbb{R})$  be as in Remark 3.1. For any  $0 < \varepsilon < 1$ ,  $0 < \varepsilon' < 1$  holds:

$$(47) \quad \text{Stab}_{[0, \varepsilon]}(\mathcal{T}) \cdot g_{\varepsilon, \varepsilon'} = \text{Stab}_{[0, \varepsilon']}(\mathcal{T}).$$

*Proof.* Using (45), (28), and the fact that  $\psi$  is diffeomorphism we compute

$$\tilde{P}_{(\sigma \cdot g_{\varepsilon, \varepsilon'})}^\mathcal{T} \cap [0, \varepsilon'] = \psi^{-1}(\tilde{P}_\sigma^\mathcal{T}) \cap \psi^{-1}([0, \varepsilon]) = \psi^{-1}(\tilde{P}_\sigma^\mathcal{T} \cap [0, \varepsilon]).$$

Now the lemma follows from the very Definition 4.3 and the property  $g_{\varepsilon, \varepsilon'}^{-1} = g_{\varepsilon', \varepsilon}$ .  $\square$

**Corollary 4.5.** Let  $\mathcal{T}$  be any triangulated category. The following are equivalent:

- (a)  $\text{Stab}_\varepsilon(\mathcal{T}) \neq \emptyset$  for some  $\varepsilon \in (0, 1)$
- (b)  $\text{Stab}_\varepsilon(\mathcal{T}) \neq \emptyset$  for each  $\varepsilon \in (0, 1)$
- (c)  $P_\sigma^\mathcal{T}$  is not dense in  $\mathbb{S}^1$  for some full  $\sigma \in \text{Stab}(\mathcal{T})$ .

*Proof.* (a)  $\Rightarrow$  (b). Follows from (46) and Lemma 4.4.

(b)  $\Rightarrow$  (c). It is obvious from the definitions that for any  $0 < \varepsilon < 1$  and any  $\sigma \in \text{Stab}_\varepsilon(\mathcal{T})$  the set  $P_\sigma^\mathcal{T}$  is not dense in  $\mathbb{S}^1$ .

(c)  $\Rightarrow$  (a). If  $P_\sigma^\mathcal{T}$  is not dense, then  $\mathbb{S}^1 \setminus P_\sigma^\mathcal{T}$  contains an open arc, but then it contains a closed arc as well and then  $\sigma \in \text{Stab}_\varepsilon(\mathcal{T})$  for some  $\varepsilon \in (0, 1)$ .  $\square$

**Definition 4.6.** A triangulated category  $\mathcal{T}$  will be called a category with phase gap if  $P_\sigma^\mathcal{T}$  is not dense in  $\mathbb{S}^1$  for some full  $\sigma \in \text{Stab}(\mathcal{T})$  (by Corollary 4.5 then  $\text{Stab}_\varepsilon(\mathcal{T})$  is not empty for any  $0 < \varepsilon < 1$ ).

**Lemma 4.7.** If  $K_0(\mathcal{T})$  has finite rank, then  $\mathcal{T}$  has a phase gap iff there exists a bounded  $t$ -structure in  $\mathcal{T}$  whose heart is of finite length and has finitely many simple objects.

*Proof.* Let  $\mathcal{A}$  be such a heart and let  $s_1, s_2, \dots, s_n$  be the simple objects in  $\mathcal{A}$ . Under the given assumptions  $K_0(\mathcal{T}) \cong K_0(\mathcal{A}) \cong \mathbb{Z}^n$ . [10, Proposition 2.4, Proposition 5.3] imply that for any sequence of vectors  $z_1, z_2, \dots, z_n$  in  $\mathbb{H}$  there exists unique stability condition  $\sigma = (Z, \mathcal{P})$  with  $\mathcal{P}(0, 1] = \mathcal{A}$  and  $Z(s_i) = z_i$ ,  $i = 1, \dots, n$ . For this  $\sigma$  we have  $Z(\mathcal{P}(0, 1] \setminus \{0\}) = \{\sum_{i=1}^n a_i z_i : (a_1, a_2, \dots, a_n) \in \mathbb{N}^n \setminus \{0\}\}$  and therefore  $Z(\sigma^{ss}) \subset \pm\{\sum_{i=1}^n a_i z_i : (a_1, a_2, \dots, a_n) \in \mathbb{N}^n \setminus \{0\}\}$ , now from [16, Lemma 1.1] it follows that  $\sigma$  is locally finite and therefore, using the notation explained after (16), we can write  $\sigma \in \mathbb{H}^\mathcal{A}$ . Varying the vector  $(z_1, z_2, \dots, z_n) \in \mathbb{H}^n$  we obtain a biholomorphism between  $\mathbb{H}^n$  and the subset  $\mathbb{H}^\mathcal{A} \subset \text{Stab}(\mathcal{T})$ . In particular the stability conditions in  $\mathbb{H}^\mathcal{A}$  are full. Since for  $\sigma \in \mathbb{H}^\mathcal{A}$  holds  $Z(\sigma^{ss}) \subset \pm\{\sum_{i=1}^n a_i z_i : (a_1, a_2, \dots, a_n) \in \mathbb{N}^n \setminus \{0\}\}$ , it follows that  $Z(\sigma^{ss}) \subset \pm\{x \exp(i\pi a) + y \exp(i\pi(a+1-\varepsilon)) : x, y \in (0, +\infty)\}$  for some  $a \in \mathbb{R}$  and some  $0 < \varepsilon < 1$ , therefore by (19) it follows that  $\sigma \in \text{Stab}_\varepsilon(\mathcal{T})$ , hence  $\mathcal{T}$  has a phase gap.

Conversely, let  $\sigma' = (Z, \mathcal{P}) \in \text{Stab}_\varepsilon(\mathcal{T})$  be a full stability condition, then due to (46) for some  $\lambda \in \mathbb{C}$  we have  $\sigma = \lambda \star \sigma'$  satisfying  $\mathcal{P}_\sigma(t) = \{0\}$  for  $t \in [0, \varepsilon]$  and  $\mathcal{P}_\sigma(0, 1] = \mathcal{P}_{\sigma'}(\varepsilon/2, 1]$ . From [11, Lemma 4.5] it follows that  $\mathcal{P}_\sigma(0, 1] = \mathcal{P}_{\sigma'}(\varepsilon/2, 1]$  is a finite length quasi-abelian category (here the property of  $\sigma$  being full is used), and since  $\sigma$  is a stability condition,  $\mathcal{P}_\sigma(0, 1]$  is a heart of a bounded  $t$ -structure, therefore it is a finite length abelian category whose simple objects are a basis of  $K_0(\mathcal{T})$ , in particular the simple objects are finitely many.  $\square$

**Remark 4.8.** The elements  $\sigma \in \text{Stab}(\mathcal{T})$  for which  $\mathcal{P}(0, 1]$  is of finite length and with finitely many simple objects are called algebraic stability conditions and have been discussed extensively in [43].

**Remark 4.9.** Due to (i), (ii) in the beginning of [12, Subsection 7.1] the CY3 categories discussed in [12] have phase gaps.

**Remark 4.10.** Let  $\mathcal{T}$  be proper and with a full exceptional collection. [20, Remark 3.20] and Corollary 4.5 imply that  $\text{Stab}_\varepsilon(\mathcal{T}) \neq \emptyset$  for any  $0 < \varepsilon < 1$ , i. e.  $\mathcal{T}$  is a category with a phase gap.

The main definition of this section is:

**Definition 4.11.** Let  $\mathcal{T}$  be a triangulated category with phase gap. Let  $0 < \varepsilon < 1$ . We define:

$$(48) \quad \|\mathcal{T}\|_\varepsilon = \sup \left\{ \frac{1}{2} \text{vol}(\overline{P_\sigma}) : \sigma \in \text{Stab}_\varepsilon(\mathcal{T}) \right\}.$$

**Remark 4.12.** For a category  $\mathcal{T}$  which carries a full stability condition, but has no phase gap (i. e.  $P_\sigma^\mathcal{T}$  is dense in  $\mathbb{S}^1$  for all full stability conditions  $\sigma$ ) it seems reasonable to define  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ , but we will restrict our attention to categories with phase gaps in the rest.

In remarks (4.13), (4.14)  $\varepsilon$  and  $\mathcal{T}$  are as in Definition 4.11.

**Remark 4.13.** Using (44), (45), (46) one shows that ( $\mu$  is the Lebesgue measure of  $\mathbb{R}$ ):

$$(49) \quad \|\mathcal{T}\|_\varepsilon = \sup \left\{ \pi \mu \left( [\varepsilon, 1] \cap \widetilde{\overline{P_\sigma}} \right) : \sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T}) \right\}.$$

**Remark 4.14.** We have always  $0 \leq \|\mathcal{T}\|_\varepsilon \leq \pi(1 - \varepsilon)$ .

**Remark 4.15.** Using Remark 4.2 we see that if  $\mathcal{T}_1, \mathcal{T}_2$  are equivalent triangulated categories with finite rank Grothendieck groups, then for any  $0 < \varepsilon < 1$  holds  $\|\mathcal{T}_1\|_\varepsilon = \|\mathcal{T}_2\|_\varepsilon$ .

**Lemma 4.16.** Let  $\varepsilon, \varepsilon'$  be any two numbers in  $(0, 1)$ .

(a) There exist  $0 < m < M$  such that  $m\|\mathcal{T}\|_\varepsilon \leq \|\mathcal{T}\|_{\varepsilon'} \leq M\|\mathcal{T}\|_\varepsilon$  for any category with a phase gap  $\mathcal{T}$ . In particular, for any category with a phase gap  $\mathcal{T}$  we have:  $\|\mathcal{T}\|_\varepsilon = 0 \iff \|\mathcal{T}\|_{\varepsilon'} = 0$ .

(b) For any category with a phase gap  $\mathcal{T}$  we have  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon) \iff \|\mathcal{T}\|_{\varepsilon'} = \pi(1 - \varepsilon')$ .

*Proof.* We will use the element  $g_{\varepsilon, \varepsilon'} = (G, \psi) \in \widetilde{GL}^+(2, \mathbb{R})$  from Remark 3.1. In particular the function  $\psi \in C^\infty(\mathbb{R})$  restricts to a diffeomorphism  $\psi| : [\varepsilon', 1] \rightarrow [\varepsilon, 1]$ . Let us denote the inverse function by  $\kappa$ , then we choose  $m, M \in \mathbb{R}$  as follows:

$$(50) \quad \psi|^{-1} = \kappa : [\varepsilon, 1] \rightarrow [\varepsilon', 1] \quad \forall t \in [\varepsilon, 1] \quad 0 < m \leq \kappa'(t) \leq M.$$

With the help of [46, formula (15) on page 156], we see that for any Lebesgue measurable subset  $A \subset [\varepsilon, 1]$  holds (for a subset  $E \subset [\varepsilon', 1]$  or  $E \subset [\varepsilon, 1]$  we denote by  $\chi_E$  the function equal to 1 at the points of  $E$  and 0 elsewhere):

$$\mu(\kappa(A)) = \int_{\varepsilon'}^1 \chi_{\kappa(A)}(t) dt = \int_\varepsilon^1 \chi_{\kappa(A)}(\kappa(t)) \kappa'(t) dt = \int_\varepsilon^1 \chi_A(t) \kappa'(t) dt$$

which by (50) implies:

$$(51) \quad m\mu(A) \leq \mu(\kappa(A)) \leq M\mu(A).$$

Using Remark 4.13, Lemma 4.4, and the second equality in (45) we get:

$$(52) \quad \|\mathcal{T}\|_\varepsilon / \pi = \sup \left\{ \mu \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) : \sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T}) \right\}$$

$$(53) \quad \|\mathcal{T}\|_{\varepsilon'} / \pi = \sup \left\{ \mu \left( \kappa \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) \right) : \sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T}) \right\}.$$

Now (a) follows from (51), (52), (53).

(b) Let  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$  and  $\delta > 0$ . We will prove that (53) equals  $(1 - \varepsilon')$  by finding  $\sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T})$  such that  $\mu \left( \kappa \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) \right) > 1 - \varepsilon' - \delta$ . Since  $1 - \varepsilon' = \mu([\varepsilon', 1]) = \mu \left( \kappa \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) \right) + \mu \left( \kappa \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) \right)$ , we need to find  $\sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T})$  such that:

$$(54) \quad \mu \left( \kappa \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) \right) < \delta.$$

Since  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ , (52) ensures that there is  $\sigma \in \text{Stab}_{[0, \varepsilon]}(\mathcal{T})$  such that  $\mu \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) > 1 - \varepsilon - \frac{\delta}{M}$ , which due to the equality  $\mu \left( [\varepsilon, 1] \cap \widetilde{P}_\sigma \right) + \mu \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) = 1 - \varepsilon$  is the same as  $\mu \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) < \frac{\delta}{M}$ . We combine (51) and the latter inequality to deduce the desired (54):  $\mu \left( \kappa \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) \right) \leq M\mu \left( [\varepsilon, 1] \setminus \widetilde{P}_\sigma \right) < \delta$ .  $\square$

[19, Corollary 3.28] (see [18, Corollary 3.25] for general algebraically closed field  $k$ ) amounts to the following criteria for non-vanishing of  $\|\mathcal{T}\|_\varepsilon$

**Proposition 4.17.** *Let  $(E_0, E_1, \dots, E_n)$  be a full exceptional collection in a  $k$ -linear proper triangulated category  $\mathcal{D}$ . If for some  $i$  the pair  $(E_i, E_{i+1})$  satisfies  $\mathrm{hom}^1(E_i, E_{i+1}) \geq 3$  and  $\mathrm{hom}^{\leq 0}(E_i, E_{i+1}) = 0$ , then  $\|\mathcal{D}\|_\varepsilon > 0$ .*

**Corollary 4.18.** *Let  $\varepsilon \in (0, 1)$ . Then:*

- (a) *If  $Q$  is an acyclic quiver, which is neither Dynkin nor affine, then  $\|D^b(Q)\|_\varepsilon > 0$ .*
- (b)  *$\|D^b(\mathrm{coh}(X))\|_\varepsilon > 0$ , where  $X$  is a smooth projective variety over  $\mathbb{C}$ , such that  $D^b(\mathrm{coh}(X))$  is generated by a strong exceptional collection of three elements*

*Proof.* (a) Follows from the previous proposition, [19, Proposition 3.34], and the fact that each exceptional collection in  $D^b(Q)$  can be extended to a full exceptional collection (see [17]).

(b) It follows from proposition 4.17 and [19, 3.5.1].  $\square$

In Section 8 we will refine Proposition 4.17, which will help us to prove that  $\|D^b(\mathrm{coh}(X))\|_\varepsilon = \pi(1 - \varepsilon)$  if  $X$  is  $\mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathbb{P}^n$  with  $n \geq 2$  or some of these blown up in finite number of points.

**Proposition 4.19.** *Let  $\varepsilon \in (0, 1)$ . For acyclic quiver  $Q$  we have  $\|D^b(Q)\|_\varepsilon = 0$  iff  $Q$  is affine or Dynkin. In particular  $\|D^b(\mathrm{coh}(\mathbb{P}^1))\|_\varepsilon = 0$ .*

*Proof.* If  $Q$  is affine or Dynkin, then from the first and the second rows of table (6) we see that  $\mathrm{vol}(\overline{P_\sigma}) = 0$  for any  $\sigma \in \mathrm{Stab}(D^b(Q))$ , therefore  $\|D^b(Q)\|_\varepsilon = 0$ , and in Corollary 4.18 we showed that  $\|D^b(Q)\|_\varepsilon > 0$  for the rest quivers.  $\square$

## 5. STABILITY CONDITIONS ON ORTHOGONAL DECOMPOSITIONS

First we recall the definition of a semi-orthogonal, resp. orthogonal, decomposition of a triangulated category:

**Definition 5.1.** *If  $\mathcal{T}$  is a triangulated category,  $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$  are triangulated subcategories in it satisfying the equalities  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$  and  $\mathrm{Hom}(\mathcal{T}_j, \mathcal{T}_i) = 0$  for  $j > i$ , then we say that  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$  is a semi-orthogonal decomposition. If in addition holds  $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0$  for  $i < j$ , then we say that  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$  is an orthogonal decomposition, in which case we will write sometimes  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \dots \oplus \mathcal{T}_n$ . Obviously, if  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$  is an orthogonal decomposition, then  $\mathcal{T} = \langle \mathcal{T}_{s(1)}, \mathcal{T}_{s(2)}, \dots, \mathcal{T}_{s(n)} \rangle$  is an orthogonal decomposition for any permutation  $s : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ .*

**Proposition 5.2.** *Let  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$  be any orthogonal decomposition. Let*

$K_0(\mathcal{T}_i) \xrightarrow{in_i} K_0(\mathcal{T}) \xrightarrow{pr_j} K_0(\mathcal{T}_j)$ ,  $1 \leq i, j \leq n$  *be the natural biproduct diagram. Then:*

(a) *The following map is a bijection:*

$$(55) \quad \mathrm{Stab}(\mathcal{T}) \rightarrow \mathrm{Stab}(\mathcal{T}_1) \times \mathrm{Stab}(\mathcal{T}_2) \times \dots \times \mathrm{Stab}(\mathcal{T}_n)$$

$$(56) \quad (Z, \{\mathcal{P}(t)\}_{t \in \mathbb{R}}) \mapsto ((Z \circ pr_1, \{\mathcal{P}(t) \cap \mathcal{T}_1\}_{t \in \mathbb{R}}), \dots, (Z \circ pr_n, \{\mathcal{P}(t) \cap \mathcal{T}_n\}_{t \in \mathbb{R}})).$$

(b) *For any  $(Z, \{\mathcal{P}(t)\}_{t \in \mathbb{R}}) \in \mathrm{Stab}(\mathcal{T})$  and any  $t \in \mathbb{R}$  the subcategory  $\mathcal{P}(t)$  is non-trivial iff for some  $j$   $\mathcal{P}(t) \cap \mathcal{T}_j$  is non-trivial.*

(c) *If  $\mathrm{rank}(K_0(\mathcal{T}_i)) < +\infty$  for all  $i = 1, 2, \dots, n$ , then the map defined above is biholomorphism.*

(d) *For each  $\sigma \in \mathrm{Stab}(\mathcal{T})$  holds  $P_\sigma^\mathcal{T} = \cup_{i=1}^n P_{\sigma_i}^{\mathcal{T}_i}$ , where  $(\sigma_1, \dots, \sigma_n)$  is the value of (55) at  $\sigma$ .*



*Proof.* We will give all details for the proof of (a), (b), (c) in the case  $n = 2$ . The general case follows easily by induction. (d) follows from the very definition (42) and (a), (b).

It is well known that for each  $X \in \mathcal{T}$  there exists unique up to isomorphism triangle  $E_2 \rightarrow X \rightarrow E_1 \rightarrow E_2[1]$  with  $E_i \in \mathcal{T}_i$ ,  $i = 1, 2$ . By  $\text{Hom}(\mathcal{T}_1, \mathcal{T}_2) = 0$  it follows that each of these triangles is actually part of a direct product diagram and  $pr_i([X]) = [E_i]$  for  $i = 1, 2$ .

Now let  $X \in \mathcal{T}_1$  and  $U \rightarrow X \rightarrow B \rightarrow U[1]$  be a triangle in  $\mathcal{T}$ . Using  $\text{Hom}(\mathcal{T}_2, \mathcal{T}_1) = \text{Hom}(\mathcal{T}_1, \mathcal{T}_2) = 0$  and decomposing  $U$  into direct summands  $U_1 \oplus U_2$  with  $U_i \in \mathcal{T}_i$  one easily concludes that the triangle  $U \rightarrow X \rightarrow B \rightarrow U[1]$  is isomorphic to a triangle of the form  $U_1 \oplus U_2 \rightarrow X \rightarrow B' \oplus U_2[1] \rightarrow U_1[1] \oplus U_2[1]$ . If we apply these arguments to the last triangle in (17) and using that  $\text{hom}(E_{n-1}, A_n[i]) = 0$  for  $i \leq 0$ , we immediately obtain  $E_{n-1}, A_n \in \mathcal{T}_1$  and then by induction it follows that the entire HN filtration of  $X$  lies in  $\mathcal{T}_1$ , in particular  $A_i \in \mathcal{P}(t_i) \cap \mathcal{T}_1$  for  $i = 1, 2, \dots, n$ , furthermore we have  $Z_1([X]) = Z(pr_1([X]))$  for each  $X \in \mathcal{P}(t) \cap \mathcal{T}_1$  and now it is obvious that  $(Z \circ pr_1, \{\mathcal{P}(t) \cap \mathcal{T}_1\}_{t \in \mathbb{R}}) = (Z_1, \mathcal{P}_1)$  is a stability condition on  $\mathcal{T}_1$ . The same arguments apply to the case  $X \in \mathcal{T}_2$  and show that  $(Z \circ pr_2, \{\mathcal{P}(t) \cap \mathcal{T}_2\}_{t \in \mathbb{R}}) = (Z_2, \mathcal{P}_2)$  is a stability condition on  $\mathcal{T}_2$ . We will show that  $\sigma_i$  are locally finite for  $i = 1, 2$ . Indeed, since  $\sigma$  is locally finite stability condition on  $\mathcal{T}$ , then there exists  $\frac{1}{2} > \varepsilon > 0$  such that  $\mathcal{P}(t - \varepsilon, t + \varepsilon)$  is quasi-abelian category of finite length for each  $t \in \mathbb{R}$ . One easily shows that  $\mathcal{P}_i(t - \varepsilon, t + \varepsilon) = \mathcal{T}_i \cap \mathcal{P}(t - \varepsilon, t + \varepsilon)$  for each  $t$ . From [10, Lemma 4.3] we know that a sequence  $A \rightarrow B \rightarrow C$  in  $\mathcal{P}_i(t - \varepsilon, t + \varepsilon)$  is a strict short exact sequence iff it is part of a triangle  $A \rightarrow B \rightarrow C \rightarrow A[1]$  in  $\mathcal{T}_i$ . Since for  $A, B, C$  in  $\mathcal{T}_i$   $A \rightarrow B \rightarrow C \rightarrow A[1]$  is triangle in  $\mathcal{T}_i$  iff it is a triangle in  $\mathcal{T}$ , we deduce that for  $A, B, C \in \mathcal{P}_i(t - \varepsilon, t + \varepsilon)$   $A \rightarrow B \rightarrow C$  is a strict exact sequence in  $\mathcal{P}_i(t - \varepsilon, t + \varepsilon)$  iff it is a strict exact sequence in  $\mathcal{P}(t - \varepsilon, t + \varepsilon)$ , and now from the fact that  $\mathcal{P}(t - \varepsilon, t + \varepsilon)$  is of finite length it follows that  $\mathcal{P}_i(t - \varepsilon, t + \varepsilon)$  is of finite length and  $\sigma_i \in \text{Stab}(\mathcal{T}_i)$  for  $i = 1, 2$ . Thus the map is well defined. Since for any interval  $I \subset \mathbb{R}$  the subcategory  $\mathcal{P}(I)$  is thick (see e.g. [18, Lemma 2.20.]), it follows that  $\mathcal{P}(t) = \mathcal{P}_1(t) \oplus \mathcal{P}_2(t)$  for each  $t \in \mathbb{R}$  and hence follows the injectivity of the map. Furthermore, using the terminology of [16, Definition before Proposition 2.2] we see that  $\sigma$  is glued from  $\sigma_1$  and  $\sigma_2$ . From the given arguments it follows also that for  $X \in \mathcal{T}_i$  the HN filtrations w.r. to  $\sigma$  and w.r. to  $\sigma_i$  coincide, in particular:

$$(57) \quad X \in \mathcal{T}_i \Rightarrow \phi_{\sigma_i}^{\pm}(X) = \phi_{\sigma}^{\pm}(X) \quad m_{\sigma_i}(X) = m_{\sigma}(X)$$

on the other hand any  $X \in \mathcal{T}$  can be represented uniquely (up to isomorphism) as a biproduct  $X \cong X_1 \oplus X_2$  with  $X_i \in \mathcal{T}_i$  for  $i = 1, 2$  and (21) imply

$$(58) \quad X \in \mathcal{T} \Rightarrow X \cong X_1 \oplus X_2, \quad X_i \in \mathcal{T}_i \Rightarrow \begin{aligned} m_{\sigma}(X) &= m_{\sigma_1}(X_1) + m_{\sigma_2}(X_2) \\ \phi_{\sigma}^{-}(X) &= \min\{\phi_{\sigma_1}^{-}(X_1), \phi_{\sigma_2}^{-}(X_2)\} \\ \phi_{\sigma}^{+}(X) &= \max\{\phi_{\sigma_1}^{+}(X_1), \phi_{\sigma_2}^{+}(X_2)\} \end{aligned}$$

Conversely, if  $(\sigma_1, \sigma_2) \in \text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$ , then [16, Proposition 3.5] ensures existence of a locally finite stability condition  $\sigma \in \text{Stab}(\mathcal{T})$  glued from  $\sigma_1, \sigma_2$  and using [16, (3) in Proposition 2.2]) one easily shows that our map sends the glued  $\sigma$  to the pair  $(\sigma_1, \sigma_2)$ , hence the surjectivity of the map follows.

Now we will show that if  $\text{rank}(K_0(\mathcal{T}_i)) < +\infty$  for  $i = 1, 2$ , then the map defined above is biholomorphism. First we show that it is continuous. We denote by  $d, d_1, d_2$  the generalized metrics on  $\text{Stab}(\mathcal{T}), \text{Stab}(\mathcal{T}_1), \text{Stab}(\mathcal{T}_2)$  (as defined in (35)). For any  $\sigma, \sigma' \in \text{Stab}(\mathcal{T})$  let  $(\sigma_1, \sigma_2)$  and  $(\sigma'_1, \sigma'_2)$  be the pairs assigned via the bijection. To show that the map is homeomorphism we

will show that :

$$(59) \quad \max\{d_1(\sigma_1, \sigma'_1), d_2(\sigma_2, \sigma'_2)\} \leq d(\sigma, \sigma')$$

$$(60) \quad d(\sigma, \sigma') \leq d_1(\sigma_1, \sigma'_1) + d_2(\sigma_2, \sigma'_2)$$

The first (59) follows easily from (57). The second requires a bit more computations, which we will present partly. Take any  $X \in \mathcal{T}$  and decompose it  $X \cong X_1 \oplus X_2$ ,  $X_i \in \mathcal{T}_i$ , then from (58) we see that

$$(61) \quad \left| \log \frac{m_\sigma(X)}{m_{\sigma'}(X)} \right| = \left| \log \frac{m_{\sigma_1}(X_1) + m_{\sigma_2}(X_2)}{m_{\sigma'_1}(X_1) + m_{\sigma'_2}(X_2)} \right| \leq \left| \log \frac{m_{\sigma_1}(X_1)}{m_{\sigma'_1}(X_1)} \right| + \left| \log \frac{m_{\sigma_2}(X_2)}{m_{\sigma'_2}(X_2)} \right|$$

$$(62) \quad \leq d_1(\sigma_1, \sigma'_1) + d_2(\sigma_2, \sigma'_2),$$

where we used, besides the definition of the generalized metrics (35), the following lemma:

**Lemma 5.3.** *For any positive real numbers  $x_1, x_2, y_1, y_2$  holds the inequality:*

$$\left| \log \frac{x_1 + x_2}{y_1 + y_2} \right| \leq \left| \log \frac{x_1}{y_1} \right| + \left| \log \frac{x_2}{y_2} \right|.$$

*Proof.* We can assume that  $\frac{x_1+x_2}{y_1+y_2} \geq 1$  (otherwise take  $\frac{y_1+y_2}{x_1+x_2}$ ). Now we consider three cases:

If  $\frac{x_1}{y_1} \geq 1$  and  $\frac{x_2}{y_2} \geq 1$ , then the desired inequality becomes  $\log \frac{x_1+x_2}{y_1+y_2} \leq \log \frac{x_1}{y_1} + \log \frac{x_2}{y_2}$  which after exponentiating is equivalent to

$$\frac{x_1 + x_2}{y_1 + y_2} \leq \frac{x_1 x_2}{y_1 y_2} \iff (x_1 + x_2)y_1 y_2 \leq x_1 x_2 (y_1 + y_2) \iff 0 \leq x_1 y_1 (x_2 - y_2) + x_2 y_2 (x_1 - y_1)$$

the latter inequality follows from  $x_1 \geq y_1, x_2 \geq y_2$ .

If  $\frac{x_1}{y_1} \leq 1$  and  $\frac{x_2}{y_2} \geq 1$ , then the desired inequality becomes  $\log \frac{x_1+x_2}{y_1+y_2} \leq \log \frac{y_1}{x_1} + \log \frac{x_2}{y_2}$  which after exponentiating is equivalent to

$$\frac{x_1 + x_2}{y_1 + y_2} \leq \frac{y_1 x_2}{x_1 y_2} \iff (x_1 + x_2)x_1 y_2 \leq y_1 x_2 (y_1 + y_2) \iff 0 \leq y_1^2 x_2 - x_1^2 y_2 + x_2 y_2 (y_1 - x_1)$$

the latter inequality follows from  $y_1 \geq x_1, x_2 \geq y_2$ .

If  $\frac{x_1}{y_1} \leq 1$  and  $\frac{x_2}{y_2} \leq 1$ , then the desired inequality becomes  $\log \frac{x_1+x_2}{y_1+y_2} \leq \log \frac{y_1}{x_1} + \log \frac{y_2}{x_2}$  which after exponentiating is equivalent to

$$\frac{x_1 + x_2}{y_1 + y_2} \leq \frac{y_1 y_2}{x_1 x_2} \iff (x_1 + x_2)x_1 x_2 \leq y_1 y_2 (y_1 + y_2) \iff 0 \leq y_1^2 y_2 - x_1^2 x_2 + y_2^2 y_1 - x_2^2 x_1$$

the latter inequality follows from  $y_1 \geq x_1, y_2 \geq x_2$ . □

Now in order to prove (60) it is enough to show that  $|\phi_\sigma^\pm(X) - \phi_{\sigma'}^\pm(X)| \leq d_1(\sigma_1, \sigma'_1) + d_2(\sigma_2, \sigma'_2)$  which in turn via (58) is the same as

$$(63) \quad \left| \max\{\phi_{\sigma_1}^+(X_1), \phi_{\sigma_2}^+(X_2)\} - \max\{\phi_{\sigma'_1}^+(X_1), \phi_{\sigma'_2}^+(X_2)\} \right| \leq d_1(\sigma_1, \sigma'_1) + d_2(\sigma_2, \sigma'_2)$$

$$(64) \quad \left| \min\{\phi_{\sigma_1}^-(X_1), \phi_{\sigma_2}^-(X_2)\} - \min\{\phi_{\sigma'_1}^-(X_1), \phi_{\sigma'_2}^-(X_2)\} \right| \leq d_1(\sigma_1, \sigma'_1) + d_2(\sigma_2, \sigma'_2),$$

which in turn follow from the following:

**Lemma 5.4.** *For any real numbers  $x_1, x_2, y_1, y_2$  we have:*

$$\begin{aligned} |\max\{x_1, x_2\} - \max\{y_1, y_2\}| &\leq |x_1 - y_1| + |x_2 - y_2| \\ |\min\{x_1, x_2\} - \min\{y_1, y_2\}| &\leq |x_1 - y_1| + |x_2 - y_2| \end{aligned}$$

*Proof.* If  $\max\{x_1, x_2\} = x_i$  and  $\max\{y_1, y_2\} = y_i$  for the same  $i$ , then the inequalities follow immediately. So let  $\max\{x_1, x_2\} = x_i$   $\max\{y_1, y_2\} = y_j$ ,  $i \neq j$ , e.g. let  $i = 1$ ,  $j = 2$ . Then  $x_1 \geq x_2$ ,  $y_1 \leq y_2$ , and the lemma follows from:

$$(65) \quad \begin{aligned} &|\max\{x_1, x_2\} - \max\{y_1, y_2\}| = |x_1 - y_2| = \\ &= \begin{cases} x_1 - y_2 = x_1 - y_1 + y_1 - y_2 \leq x_1 - y_1 = |x_1 - y_1| & \text{if } x_1 \geq y_2 \\ y_2 - x_1 = y_2 - x_2 + x_2 - x_1 \leq y_2 - x_2 = |x_2 - y_2| & \text{if } x_1 \leq y_2 \end{cases} \end{aligned}$$

$$(66) \quad \begin{aligned} &|\min\{x_1, x_2\} - \min\{y_1, y_2\}| = |x_2 - y_1| = \\ &= \begin{cases} x_2 - y_1 = x_2 - x_1 + x_1 - y_1 \leq x_1 - y_1 = |x_1 - y_1| & \text{if } x_2 \geq y_1 \\ y_1 - x_2 = y_1 - y_2 + y_2 - x_2 \leq y_2 - x_2 = |x_2 - y_2| & \text{if } x_2 \leq y_1 \end{cases} . \end{aligned}$$

□

Thus, we have (59), (60) and they imply that (55) is homeomorphism for  $n = 2$ .

Let  $\text{Stab}(\mathcal{T}) \xrightarrow{\text{proj}} \text{Hom}(K_0(\mathcal{T}), \mathbb{C})$ ,  $\text{Stab}(\mathcal{T}_i) \xrightarrow{\text{proj}_i} \text{Hom}(K_0(\mathcal{T}_i), \mathbb{C})$ ,  $i = 1, 2$  be the projections  $\text{proj}(Z, \mathcal{P}) = Z$ . Then the following diagram (the first row is the map (55) and the second row is the assignment  $Z \mapsto (Z \circ \text{pr}_1, Z \circ \text{pr}_2)$ ) is commutative:

$$\begin{array}{ccc} \text{Stab}(\mathcal{T}) & \xrightarrow{\varphi} & \text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2) \\ \text{proj} \downarrow & & \text{proj}_1 \times \text{proj}_2 \downarrow \\ \text{Hom}(K_0(\mathcal{T}), \mathbb{C}) & \xrightarrow{\varphi'} & \text{Hom}(K_0(\mathcal{T}_1), \mathbb{C}) \times \text{Hom}(K_0(\mathcal{T}_2), \mathbb{C}). \end{array}$$

If we take any connected component  $\Sigma \subset \text{Stab}(\mathcal{T})$ , then (since  $\varphi$  is homeomorphism)  $\varphi(\Sigma) = \Sigma_1 \times \Sigma_2$  is a connected component of  $\text{Stab}(\mathcal{T}_1) \times \text{Stab}(\mathcal{T}_2)$ , resp.  $\Sigma_i$  are connected components of  $\text{Stab}(\mathcal{T}_i)$ , and furthermore  $m = \dim_{\mathbb{C}}(\Sigma) = \dim_{\mathbb{C}}(\Sigma_1 \times \Sigma_2)$ . From the Bridgeland's main theorem we know that  $\text{proj}$  restricts to local biholomorphisms between  $\Sigma$  and an  $m$ -dimensional vector subspace  $V \subset \text{Hom}(K_0(\mathcal{T}), \mathbb{C})$  and  $\text{proj}_1 \times \text{proj}_2$  restricts to local biholomorphisms between  $\Sigma_1 \times \Sigma_2$  and an  $m$ -dimensional vector subspace  $V_1 \times V_2 \subset \text{Hom}(K_0(\mathcal{T}_1), \mathbb{C}) \times \text{Hom}(K_0(\mathcal{T}_2), \mathbb{C})$ . It follows (using that  $\varphi'$  is a linear isomorphism and that each open subset in a vector subset contains a basis of the space) that  $\varphi'(V) = V_1 \times V_2$ . Thus, the diagram above restricts to a diagram with vertical arrows which are local biholomorphisms, the bottom arrow is biholomorphism, and the top arrow is a homeomorphism, it follows with standard arguments that the top arrow must be biholomorphic. It follows that  $\varphi$  is biholomorphism and we proved the proposition. □

From this proposition and Definition 4.3 it follows:

**Corollary 5.5.** *Let  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \dots \oplus \mathcal{T}_n$  be an orthogonal decomposition (Definition 5.1) and let  $\text{rank}(K_0(\mathcal{T}_i)) < +\infty$  for  $i = 1, \dots, n$ . Let  $\text{Stab}(\mathcal{T}) \rightarrow \text{Stab}(\mathcal{T}_1) \times \dots \times \text{Stab}(\mathcal{T}_n)$ ,  $\sigma \mapsto (\sigma_1, \sigma_2, \dots, \sigma_n)$  be the biholomorphism from Proposition 5.2 (a).*

*For any  $0 < \varepsilon < 1$  the following are equivalent: (a)  $\sigma \in \text{Stab}_{\varepsilon}(\mathcal{T})$ ; (b)  $\{\sigma_i \in \text{Stab}_{\varepsilon}(\mathcal{T}_i)\}_{i=1}^n$  and there exists a closed  $\varepsilon$ -arc  $\gamma$  such that  $P_{\sigma_i}^{\mathcal{T}_i} \cap \gamma = \emptyset$  for each  $1 \leq i \leq n$ .*

*In particular  $\mathcal{T}$  has a phase gap iff  $\mathcal{T}_i$  has a phase gap for each  $1 \leq i \leq n$ .*

Since the closure of  $A \cup B$  equals the union of closures of  $A$  and  $B$  and  $\text{vol}(A) \leq \text{vol}(A \cup B) \leq \text{vol}(A) + \text{vol}(B)$ , from Corollary 5.5 it follows:

**Corollary 5.6.** *Let  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_n$  be an orthogonal decomposition with finite rank Grothendieck groups of the factors, and let  $0 < \varepsilon < 1$ .*

*If  $\mathcal{T}$  has a phase gap and  $\|\mathcal{T}_j\|_\varepsilon = 0$  for some  $j$ , then  $\|\mathcal{T}\|_\varepsilon = \|\langle \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{j-1}, \mathcal{T}_{j+1}, \dots, \mathcal{T}_n \rangle\|_\varepsilon$ .*

## 6. THE INEQUALITY $\|\langle \mathcal{T}_1, \mathcal{T}_2 \rangle\|_\varepsilon \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\}$

Here we show conditions which ensure  $\|\langle \mathcal{T}_1, \mathcal{T}_2 \rangle\|_\varepsilon \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\}$  for any  $\varepsilon \in (0, 1)$ , where  $\langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  is a semi-orthogonal decomposition (see Definition 5.1) of some  $\mathcal{T}$ .

**Theorem 6.1.** *Let  $\mathcal{T}$  be proper and let  $K_0(\mathcal{T})$  has finite rank. Assume  $0 < \varepsilon < 1$ . Let  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  be a semi-orthogonal decomposition. If  $\mathcal{T}_1, \mathcal{T}_2$  are categories with phase gaps, then  $\mathcal{T}$  is a category with phase gap and for any  $0 < \varepsilon < 1$  holds :*

$$(67) \quad \|\langle \mathcal{T}_1, \mathcal{T}_2 \rangle\|_\varepsilon \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\}.$$

*Proof.* Take any  $0 < \mu$ . Let  $\sigma_i = (Z_i, \mathcal{P}_i) \in \text{Stab}_\varepsilon(\mathcal{T}_i)$  be full stability conditions, s. t.  $\frac{\text{vol}(\overline{P_{\sigma_i}^{\mathcal{T}_i}})}{2} > \|\mathcal{T}_i\|_\varepsilon - \mu$  for  $i = 1, 2$ . Due to (46) we can assume that  $\exp(i\pi[0, \varepsilon]) \subset \mathbb{S}^1 \setminus P_{\sigma_i}^{\mathcal{T}_i}$ . By the same arguments as in the last paragraph of the proof of Lemma 4.7 it follows that  $\mathcal{P}_{\sigma_i}(0, 1]$  are finite length abelian categories, therefore the simple objects in them are a basis of  $K_0(\mathcal{T}_i)$  for  $i = 1, 2$ , and these abelian categories are the extension closures of their simple objects. In particular the sets of simple objects are finite and it follows that for some  $j \in \mathbb{Z}$   $\text{Hom}^{\leq 1}(\mathcal{P}_{\sigma_1}(0, 1], \mathcal{P}_{\sigma_2}(0, 1][j]) = \text{Hom}^{\leq 1}(\mathcal{P}_{\sigma_1}(0, 1], \mathcal{P}_{\sigma_2}(j, j+1]) = 0$ . Recalling (31) we deduce that  $\text{Hom}^{\leq 1}(\mathcal{P}_{\sigma_1}(0, 1], \mathcal{P}_{(-ij\pi)\star\sigma_2}(0, 0+1]) = 0$ . By replacing  $\sigma_2$  with  $(-ij\pi) \star \sigma_2$  we obtain stability conditions  $\sigma_i \in \text{Stab}_\varepsilon(\mathcal{T}_i)$  for  $i = 1, 2$  satisfying the following conditions:

$$(68) \quad \frac{\text{vol}(\overline{P_{\sigma_i}^{\mathcal{T}_i}})}{2} > \|\mathcal{T}_i\|_\varepsilon - \mu \quad \text{for } i = 1, 2,$$

$$(69) \quad \text{Hom}^{\leq 1}(\mathcal{P}_{\sigma_1}(0, 1], \mathcal{P}_{\sigma_2}(0, 1]) = 0,$$

$$(70) \quad \mathcal{P}_{\sigma_2}(0, 1] \quad \text{and} \quad \mathcal{P}_{\sigma_2}(0, 1] \quad \text{are of finite length and with f.m. simples,}$$

$$(71) \quad \mathcal{P}_{\sigma_i}(t) = \{0\} \quad \text{for } t \in [j, j + \varepsilon] \quad \text{for } i = 1, 2, j \in \mathbb{Z}.$$

In the listed properties of  $\sigma_i \in \text{Stab}(\mathcal{T}_i)$  with the given semi-orthogonal decomposition  $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$  are contained the conditions of [16, Proposition 3.5 (b)]. This proposition ensures a glued (see [16, Definition ]) locally finite stability condition  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$ . The glued stability condition satisfies the following (we use [16, Proposition 2.2 (3)] and write  $\mathcal{P}_i$  instead of  $\mathcal{P}_{\sigma_i}$ )

$$(72) \quad \mathcal{P}(0, 1] \quad \text{is extension closure of } \mathcal{P}_1(0, 1], \mathcal{P}_2(0, 1]$$

$$(73) \quad \forall i \in \{1, 2\} \quad \forall t \in \mathbb{R} \quad \mathcal{P}_i(t) \subset \mathcal{P}(t)$$

$$(74) \quad Z(X) = Z_1(X) \quad \text{for } X \in \mathcal{T}_1; \quad Z(X) = Z_2(X) \quad \text{for } X \in \mathcal{T}_2.$$

We will show that

$$(75) \quad t \in [0, \varepsilon] \Rightarrow \mathcal{P}(t) = 0.$$

Indeed, let  $s_{11}, s_{12}, \dots, s_{1n}$  and  $s_{21}, s_{22}, \dots, s_{2m}$  be the simple objects of  $\mathcal{P}_1(0, 1]$  and  $\mathcal{P}_2(0, 1]$ , respectively. Then  $\{s_{1i}\}_{i=1}^n \subset \sigma_1^{ss}$ ,  $\{s_{2i}\}_{i=1}^m \subset \sigma_2^{ss}$  and by (71), (74), and (19) we deduce that

$$(76) \quad Z(s_{1i}), Z(s_{2j}) \in \mathbb{R}_{>0} \exp(i\pi(\varepsilon, 1)),$$

and on the other hand by (72) it follows that  $Z(X)$  is a positive linear combination of  $\{Z(s_{1i})\}_{i=1}^n$ ,  $\{Z(s_{2i})\}_{i=1}^m$  for  $X \in \mathcal{P}(t) \setminus \{0\}$ ,  $t \in (0, 1]$ , and therefore  $Z(X) \in \mathbb{R}_{>0} \exp(i\pi(\varepsilon, 1))$ , hence (19) gives  $\phi_\sigma(X) \in (\varepsilon, 1)$  and (75) follows. This in turn implies  $\exp(i\pi[0, \varepsilon]) \cap P_\sigma^\mathcal{T} = \emptyset$  and then for obtaining  $\sigma \in \text{Stab}_\varepsilon(\mathcal{T})$  (recall Definition 4.3) it remains to show that  $\sigma$  is a full stability condition. We will prove this by showing that  $\mathcal{P}(0, 1]$  is a finite length abelian category (then it follows that  $\mathbb{H}^{\mathcal{P}(0,1]} \cong \mathbb{H}^{n+m}$  and  $\sigma$  is full, since  $\sigma \in \mathbb{H}^{\mathcal{P}(0,1]}$ ). However [16, Proposition 3.5 (a)] claims that if 0 is an isolated point for  $\mathfrak{S}(Z_i(\mathcal{P}_i(0, 1]))$  for  $i = 1, 2$  (which is satisfied due to (70) and (71)), then  $\mathcal{P}(0, 1]$  is a finite length category, and on the other hand due to (75) holds  $\mathcal{P}(0, 1] = \mathcal{P}(0, 1)$ . Therefore indeed  $\mathcal{P}(0, 1]$  is finite length category and  $\sigma$  is a full stability condition.

Finally, from (73) it follows that  $P_{\sigma_i}^{\mathcal{T}_i} \subset P_\sigma^\mathcal{T}$ , therefore  $\overline{P_{\sigma_i}^{\mathcal{T}_i}} \subset \overline{P_\sigma^\mathcal{T}}$ , and hence  $\text{vol}(\overline{P_{\sigma_i}^{\mathcal{T}_i}}) \leq \text{vol}(\overline{P_\sigma^\mathcal{T}})$  for  $i = 1, 2$ , recalling (68) we derive:

$$(77) \quad \frac{\text{vol}(\overline{P_\sigma^\mathcal{T}})}{2} \geq \max\{\|\mathcal{T}_1\|_\varepsilon, \|\mathcal{T}_2\|_\varepsilon\} - \mu.$$

This inequality holds for any  $\mu > 0$  and from the very definition 4.11 we deduce (67).  $\square$

**Corollary 6.2.** *For any exceptional collection  $(E_0, E_1, \dots, E_n)$  in a proper triangulated category and for any  $0 \leq i \leq n$  we have:*

$$(78) \quad \|\langle E_0, E_1, \dots, E_n \rangle\|_\varepsilon \geq \max\{\|\langle E_0, E_1, \dots, E_i \rangle\|_\varepsilon, \|\langle E_{i+1}, E_{i+2}, \dots, E_n \rangle\|_\varepsilon\}.$$

*Proof.* Due to Remark 4.10 the categories  $\langle E_0, E_1, \dots, E_n \rangle$ ,  $\langle E_0, E_1, \dots, E_i \rangle$ ,  $\langle E_{i+1}, E_{i+2}, \dots, E_n \rangle$  have phase gaps. All the conditions of Theorem 6.1 are satisfied for the semi-orthogonal decomposition  $\langle E_0, E_1, \dots, E_n \rangle = \langle \langle E_0, E_1, \dots, E_i \rangle, \langle E_{i+1}, E_{i+2}, \dots, E_n \rangle \rangle$ , hence equality (67) gives rise to (78).  $\square$

**Corollary 6.3.** *Let  $X$  be a smooth algebraic variety and let  $Y$  be a smooth subvariety so that  $K_0(D^b(X))$ ,  $K_0(D^b(Y))$  have finite rank and  $D^b(X)$ ,  $D^b(Y)$  have phase gaps. Denote by  $\tilde{X}$  the smooth algebraic variety obtained by blowing up  $X$  along the center  $Y$ .*

*Then  $D^b(\tilde{X})$  has phase gap and  $\|D^b(\tilde{X})\|_\varepsilon \geq \max\{\|D^b(X)\|_\varepsilon, \|D^b(Y)\|_\varepsilon\}$  for any  $\varepsilon \in (0, 1)$ .*

*Proof.* [8, Theorem 4.2] ensures that there is a semi-orthogonal decomposition

$D^b(\tilde{X}) = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k, D^b(X) \rangle$ , where  $\mathcal{T}_i$  is equivalent to  $D^b(Y)$  for  $i = 1, 2, \dots, k$ . Now Theorem 6.1 ensures that the inequality holds.  $\square$

## 7. THE SPACE OF STABILITY CONDITIONS AND THE NORMS ON WILD KRONECKER QUIVERS

This section is devoted to our main example (namely  $D^b(K(l))$ ), where we compute both: the stability space and  $\|\cdot\|_\varepsilon$ .

**7.1. Recollection on the action of  $\mathrm{SL}(2, \mathbb{Z})$  on  $\mathcal{H}$ .** Let  $\mathrm{GL}^+(2, \mathbb{R})$  be the group of  $2 \times 2$  matrices with positive determinant. Recall that (see e.g. [15]) for any matrix  $\gamma \in \mathrm{GL}^+(2, \mathbb{R})$  we have a biholomorphism:

$$(79) \quad \mathrm{GL}^+(2, \mathbb{R}) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha : \mathcal{H} \rightarrow \mathcal{H} \quad \alpha(z) = \frac{az + b}{cz + d},$$

and this defines an action of  $\mathrm{GL}^+(2, \mathbb{R})$  on  $\mathcal{H}$ . Let  $\mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{GL}^+(2, \mathbb{R})$  be the subgroup of matrices with integer coefficients and determinant 1. The action of  $\mathrm{SL}(2, \mathbb{Z})$  on  $\mathcal{H}$  defined by the formula (79) is properly discontinuous (see e.g. [38, p.17 and p.20]).<sup>8</sup>

**7.1.1. Hyperbolic, parabolic, elliptic elements.** An element  $\alpha \in \mathrm{GL}^+(2, \mathbb{R})$  is called *elliptic*, *parabolic*, *hyperbolic*, if  $\mathrm{tr}(\alpha)^2 < 4\det(\alpha)$ ,  $\mathrm{tr}(\alpha)^2 = 4\det(\alpha)$ ,  $\mathrm{tr}(\alpha)^2 > 4\det(\alpha)$ , respectively. When  $\alpha$  is parabolic or hyperbolic, then  $\alpha$  has no fixed points in  $\mathcal{H}$  (see [38, p. 7]). Furthermore, if  $\alpha$  is elliptic, parabolic, or hyperbolic and  $\alpha \in \mathrm{SL}(2, \mathbb{Z})$ , then each non-trivial element in the subgroup  $\langle \alpha \rangle$  generated by  $\alpha$  is elliptic, parabolic, or hyperbolic, respectively.

**7.1.2. Fundamental domain of a hyperbolic element.** The definition of a fundamental domain in  $\mathcal{H}$  of a subgroup  $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$  which we adopt here is in [38, p. 20]. We will need to determine fundamental domains of subgroups of the form  $\langle \alpha \rangle$  for a non-scalar  $\alpha \in \mathrm{SL}(2, \mathbb{Z})$ . The following arguments will be useful.

Let  $\alpha \in \mathrm{SL}(2, \mathbb{Z})$  be a hyperbolic element. It is well known that  $\alpha$  is conjugate (in  $\mathrm{GL}^+(2, \mathbb{R})$ ) to a matrix of the form (80), i.e.  $\alpha' = \beta^{-1}\alpha\beta$  for some  $\beta$  (see e.g. [38, Lemma 1.3.2]):

$$(80) \quad \alpha' = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \quad ad > 0 \quad \alpha'(z) = (a/d)z.$$

It is clear that any strip  $F'$  as in Figure (1a), where  $\delta > 0$ , is a fundamental domain for the subgroup  $\langle \alpha' \rangle$ . Two points in  $F'$  lie in common orbit iff they are of the form  $z_- \in \mathfrak{b}'_-$ ,  $z_+ = (a/d)z_-$ . Transforming  $F'$  via  $\beta$  results in a fundamental domain  $F$  for  $\langle \alpha \rangle$ , i. e.  $F = \beta(F')$ .

**Remark 7.1.** For computing  $\beta(F')$  we will use the feature of  $\beta$  that it maps circles or lines perpendicular to the real axis to circles or lines perpendicular to the real axis (see [38, Lemma 1.1.1] and recall that  $\beta$  is a conformal map, which maps the real axis into itself).

**7.1.3. Fundamental domain of a parabolic element.** If  $\alpha \in \mathrm{SL}(2, \mathbb{Z})$  is parabolic, then  $\alpha$  is conjugate to a matrix of the form (81), i.e.  $\alpha' = \beta^{-1}\alpha\beta$  for some  $\beta \in \mathrm{GL}^+(2, \mathbb{R})$  ([38, Lemma 1.3.2]):

$$(81) \quad \alpha' = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \quad a \neq 0 \quad \alpha'(z) = z + b/a.$$

Any strip  $F'$  as in Figure (1b), where  $\delta \in \mathbb{R}$ , is a fundamental domain for subgroup  $\langle \alpha' \rangle$ . Two points in  $F'$  lie in a common orbit iff they are of the form  $z_- \in \mathfrak{b}'_-$ ,  $z_+ = z_- + |b/a|$ . Transforming  $F'$  via  $\beta$  results in a fundamental domain  $F$  for  $\langle \alpha \rangle$ , i. e.  $F = \beta(F')$ .

**Remark 7.2.** Let  $\alpha \in \mathrm{SL}(2, \mathbb{Z})$  be a non-scalar matrix. From the arguments in Subsections 7.1.3, 7.1.2 it follows that if  $\alpha$  is either hyperbolic or parabolic, then we have:

- (a)  $\langle \alpha \rangle \cong \mathbb{Z}$  acts free and properly discontinuous on  $\mathcal{H}$ ;
- (b) Let  $F$  be a fundamental domain of  $\langle \alpha \rangle$  obtained by the method in Subsections 7.1.2, 7.1.3. Then for each  $u \in \mathrm{Bd}_{\mathcal{H}}(F)$  there exists an open subset  $U \subset \mathcal{H}$ , s. t.  $u \in U$  and

<sup>8</sup>The definition of properly discontinuous can be seen in the proof of Corollary 7.18.

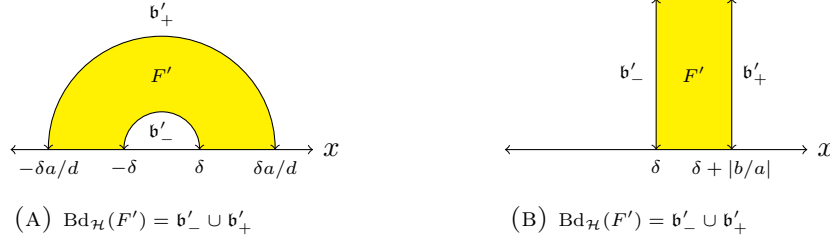


FIGURE 1. Fundamental domains

$\{i \in \mathbb{Z} : \alpha^i(U) \cap F \neq \emptyset\}$  is finite (in fact contains only two elements).

**7.2. The Helix in  $D^b(\text{Rep}_k(K(l)))$  for  $l \geq 2$ .** From now on we assume that  $l \geq 2$  and denote  $\mathcal{T}_l = D^b(\text{Rep}_k(K(l)))$ . On [35, p. 668 down] Macrì constructs a family of exceptional objects  $\{s_i\}_{i \in \mathbb{Z}}$  and states that: if  $i < j$ , then  $\text{hom}^k(s_i, s_j) \neq 0$  only if  $k = 0$ , and  $\text{hom}^k(s_j, s_i) \neq 0$  only if  $k = 1$ , without giving a proof of this statement. In this section we view  $\{s_i\}_{i \in \mathbb{Z}}$  as a helix, as defined in [9, p. 222] (and introduced earlier in [6], [28]) and using the results about geometric helices in [9, p. 222] we give a simple proof of Lemma 7.5.

Lemma 7.4 is the place where the  $\text{SL}(2, \mathbb{Z})$ -action on  $\mathcal{H}$  comes into play, it is a simple but important observation for the rest of the paper.

We write  $\underline{\dim}(X) = (n, m)$ ,  $\underline{\dim}_0(X) = n$ ,  $\underline{\dim}_1(X) = m$ . for a representation of the form:

$$X = \begin{matrix} k^n & & k^m \\ & \curvearrowright & \\ & \vdots & \\ & \curvearrowleft & \end{matrix} \in \text{Rep}_k(K(l))$$

Recall that  $\text{Rep}_k(K(l))$  is hereditary category in which for any two  $X, Y \in \text{Rep}_k(Q)$  with dimension vectors  $\underline{\dim}(X) = (n_x, m_x)$ ,  $\underline{\dim}(Y) = (n_y, m_y)$  holds the equality (the Euler Formula):

$$(82) \quad \text{hom}(X, Y) - \text{hom}^1(X, Y) = n_x n_y + m_x m_y - l n_x m_y$$

Let  $s_0, s_1 \in \mathcal{T}_l$  be so that  $s_0[1], s_1$  are the simple objects in  $\text{Rep}_k(Q)$  with  $\underline{\dim}(s_0[1]) = (1, 0)$  and  $\underline{\dim}(s_1) = (0, 1)$ . Using (82) one easily computes  $\text{hom}(s_0, s_1) = l$ ,  $\text{hom}^p(s_0, s_1) = 0$  for  $p \neq 0$  and  $\text{hom}^*(s_1, s_0) = 0$ . With the terminology from Section 2 we can say that  $(s_0, s_1)$  is a strong exceptional pair, furthermore it is a full exceptional pair in  $\mathcal{T}_l = D^b(\text{Rep}_k(K(l)))$ .

**Remark 7.3.** Recall that (see e.g. [9, p. 222]) for any exceptional pair  $(A, B)$  in any proper triangulated category  $\mathcal{T}$  one defines objects  $L_A(B)$  (left mutation) and  $R_B(A)$  (right mutation) by the triangles

$$(83) \quad L_A(B) \longrightarrow \text{Hom}^*(A, B) \otimes A \xrightarrow{ev_{A,B}^*} B \quad A \xrightarrow{coev_{A,B}^*} \text{Hom}^*(A, B)^\vee \otimes B \longrightarrow R_B(A)$$

and  $(L_A(B), A), (B, R_B(A))$  are exceptional pairs as well, they are full if  $(A, B)$  is full.

It follows, that for any exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  in  $\mathcal{T}$  and for any  $0 \leq i < n$  the sequences  $R_i(\mathcal{E}) = (E_0, E_1, \dots, E_{i+1}, R_{E_{i+1}}(E_i), \dots, E_n)$ ,  $L_i(\mathcal{E}) = (E_0, E_1, \dots, L_{E_i}(E_{i+1}), E_i, \dots, E_n)$  are exceptional and  $\langle R_i(\mathcal{E}) \rangle = \langle L_i(\mathcal{E}) \rangle = \langle \mathcal{E} \rangle$ . The sequences  $L_i(\mathcal{E})$  and  $R_i(\mathcal{E})$  are called left and right mutations of  $\mathcal{E}$ .

In particular, from the exceptional pair  $(s_0, s_1)$  we get objects  $L_{s_0}(s_1)$ ,  $R_{s_1}(s_0)$ , denoted by  $s_{-1}$ ,  $s_2$ , respectively, and each two adjacent elements in the sequence  $s_{-1}, s_0, s_1, s_2$  form a full exceptional pair. Applying iteratively left/right mutations on the left/right standing exceptional pair generates

a sequence (infinite in both directions) of exceptional objects  $\{s_i\}_{i \in \mathbb{Z}}$ . This is the helix induced by the exceptional pair  $(s_0, s_1)$ , as defined in [9, p. 222]. Any two adjacent elements in this sequence form a full exceptional pair in  $\mathcal{T}$ . Actually the right mutation generates an action of  $\mathbb{Z}$  on the set of equivalence classes of exceptional pairs (w.r.  $\sim$ ), (the inverse is the left mutation). By the transitivity of this action shown in [17], it follows that in  $\mathcal{T}_l$ :

(84) *complete lists of exceptional pairs and objects (up to shifts) are  $\{(s_i, s_{i+1})\}_{i \in \mathbb{Z}}$  and  $\{s_i\}_{i \in \mathbb{Z}}$ .*

It is shown in [9, Example 2.7] that any strong exceptional pair is geometric ([9, Definition on p. 223]), which applied to our strong exceptional collection  $(s_0, s_1)$  gives the following vanishings:

$$(85) \quad i \leq j, p \neq 0 \quad \Rightarrow \quad \text{hom}^p(s_i, s_j) = 0.$$

We will show in Lemma 7.5 that, when  $p = 0$ , the dimensions above do not vanish. When  $j = i + 1$ , we use the equalities  $\text{hom}^p(L_A(B), A) = \text{hom}^{-p}(A, B)$  for any  $p \in \mathbb{Z}$ , and analogously for  $R_B(A)$ , (see again [9, Example 2.7], or [19, p. 157 down] for details) to deduce that  $\text{hom}(s_i, s_{i+1}) = \text{hom}(s_0, s_1) = l$  for any  $i$ . We obtained  $\{s_i\}_{i \in \mathbb{Z}}$  using the triangles (83), therefore for any  $i \in \mathbb{Z}$  exists a triangle in  $\mathcal{T}$ :

$$(86) \quad s_{i-1} \rightarrow s_i^l \rightarrow s_{i+1} \rightarrow s_{i-1}[1].$$

Using these triangles we prove:

**Lemma 7.4.** *Let  $i \in \mathbb{Z}$  be any integer. Let  $Z : K_0(\mathcal{T}_l) \rightarrow \mathbb{C}$  be any group homomorphism, s.t.  $\frac{Z(s_i)}{Z(s_{i-1})} \in \mathcal{H}$ . Then for any  $j \in \mathbb{Z}$  we have  $\frac{Z(s_{i+j})}{Z(s_{i-1+j})} \in \mathcal{H}$  and the equality:*

$$(87) \quad \frac{Z(s_{i+j})}{Z(s_{i-1+j})} = \alpha_l^j \left( \frac{Z(s_i)}{Z(s_{i-1})} \right),$$

where  $\alpha_l = \begin{bmatrix} l & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$  and  $\alpha_l^j : \mathcal{H} \rightarrow \mathcal{H}$  is the corresponding automorphism given in (79).

*Proof.* From (86) it follows that  $Z(s_{i+1}) = lZ(s_i) - Z(s_{i-1})$  for any  $i \in \mathbb{Z}$ , hence we can write:

$$(88) \quad \begin{bmatrix} Z(s_{i+1}) \\ Z(s_i) \end{bmatrix} = \alpha_l \begin{bmatrix} Z(s_i) \\ Z(s_{i-1}) \end{bmatrix}; \quad \begin{bmatrix} Z(s_i) \\ Z(s_{i-1}) \end{bmatrix} = \alpha_l^{-1} \begin{bmatrix} Z(s_{i+1}) \\ Z(s_i) \end{bmatrix}, \quad \text{where } \alpha_l = \begin{bmatrix} l & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$$

hence by induction we obtain:

$$(89) \quad \begin{bmatrix} Z(s_{i+j}) \\ Z(s_{i-1+j}) \end{bmatrix} = \begin{bmatrix} l & -1 \\ 1 & 0 \end{bmatrix}^j \begin{bmatrix} Z(s_i) \\ Z(s_{i-1}) \end{bmatrix} \quad j \in \mathbb{Z},$$

therefore  $\frac{Z(s_{i+j})}{Z(s_{i-1+j})} = \frac{(\alpha_l^j)_{11}Z(s_i) + (\alpha_l^j)_{12}Z(s_{i-1})}{(\alpha_l^j)_{21}Z(s_i) + (\alpha_l^j)_{22}Z(s_{i-1})}$  and, recalling (79), we deduce (87).  $\square$

The statement of the following lemma is a slight modification of [35, Lemma 4.1] and we give a proof here:



**Lemma 7.5.** *Assume that  $l \geq 2$ . Then no two elements in  $\{s_i\}_{i \in \mathbb{Z}}$  are isomorphic and:  $s_{\leq 0}[1], s_{\geq 1} \in \text{Rep}_k(K(l))$ . Furthermore, we have:*

$$(90) \quad i \leq j \quad \Rightarrow \quad \begin{cases} \text{hom}(s_i, s_j) \neq 0 \\ \text{hom}^p(s_i, s_j) = 0 \end{cases} \quad \text{for } p \neq 0 \quad ;$$

$$(91) \quad i > j + 1 \quad \Rightarrow \quad \begin{cases} \text{hom}^1(s_i, s_j) \neq 0 \\ \text{hom}^p(s_i, s_j) = 0 \end{cases} \quad \text{for } p \neq 1 \quad .$$

*Proof.* The matrix  $\alpha_l$  in (87) has trace  $\text{tr}(\alpha_l) = l \geq 2$  and therefore (see Subsection 7.1.1)  $\alpha_l^j$  is either parabolic or hyperbolic for any  $j \in \mathbb{Z}$ , in particular it has no fixed points in  $\mathcal{H}$ . If  $s_i \cong s_{i+j}$  for some  $i \in \mathbb{Z}, j \in \mathbb{N}$ , then since  $(s_{i-1}, s_i)$  and  $(s_{i+j-1}, s_{i+j})$  are both full exceptional pairs it follows that  $s_{i-1} \cong s_{i+j-1}$  (note that  $s_{i-1} \cong s_{i+j-1}[k]$  for  $k \neq 0$  is impossible by the already proved (85)), and hence  $\frac{Z(s_{i+j})}{Z(s_{i-1+j})} = \frac{Z(s_i)}{Z(s_{i-1})}$ , which contradicts (87). Therefore no two elements in  $\{s_i\}_{i \in \mathbb{Z}}$  are isomorphic, and the non-vanishings in (90) follow. Indeed, if  $\text{hom}(s_i, s_j) = 0$  for some  $i < j$ , then  $(s_j, s_i)$  is a full exceptional pair, which contradicts the already proven  $s_{i-1} \not\cong s_j$ . Since  $\text{Rep}_k(K(l))$  is hereditary category, the non-vanishings in (90) imply that  $s_{\leq 0}[1], s_{\geq 1} \in \text{Rep}_k(K(l))$ .

Recall [7] that there exists an exact functor (the Serre functor)  $F : \mathcal{T}_l \rightarrow \mathcal{T}_l$ , s. t.  $\text{hom}(A, B) = \text{hom}(B, F(A))$  for any two objects  $A, B \in \mathcal{T}$ . Furthermore, the formula on [9, p.223 above] (in our case  $n = 2$ ) says that  $F(s_i) \cong s_{i-2}[1]$  for each  $i \in \mathbb{Z}$ , hence for any three integers  $i, j, p$  we have  $\text{hom}^p(s_i, s_j) = \text{hom}^{1-p}(s_j, s_{i-2})$ . Now (91) follows from (90).  $\square$

The following corollary of [37, Theorem 0.1] ensures existence of a functor in  $\text{Aut}(\mathcal{T}_l)$ , which plays the role of the functor  $(\cdot) \otimes \mathcal{O}(1)$  for any  $l \in \mathbb{Z}$ .

**Corollary 7.6.** (of [37, Theorem 0.1]) *There exists  $A_l \in \text{Aut}(\mathcal{T}_l)$ , s. t.  $A_l(s_i) \cong s_{i+1}$  for each  $i \in \mathbb{Z}$  (recall that  $l \geq 2$ ).*

*Proof.* In [37, Definitions 2.2, 2.3], from any quiver  $Q$  they define a quiver, denoted by  $\mathbf{\Gamma}^{irr}$ , which is the disjoint union of all connected components of the Auslander-Reiten quiver of  $D^b(Q)$ , isomorphic to the connected component which contains the simple representations of  $Q$ . [37, Theorem 2.4] applied to the quiver  $Q = K(l)$  gives an isomorphism of quivers  $\rho : \mathbb{Z} \times (\mathbb{Z}K(l)) \rightarrow \mathbf{\Gamma}^{irr}$ , where:

(a)  $\mathbb{Z}K(l)$  is isomorphic to quiver with set of vertices  $\mathbb{Z}$  and for any  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  there are  $l$  arrows from  $i$  to  $j$  iff  $j = i + 1$  and no arrows otherwise;

(b)  $\mathbb{Z} \times (\mathbb{Z}K(l))$  is quiver whose connected components are  $\{i\} \times \mathbb{Z}K(l)$  and each of them is a labeled copy of  $\mathbb{Z}K(l)$ .

(c) The translation functor of  $\mathcal{T}_l$  induces action on  $\mathbf{\Gamma}^{irr}$  which by  $\rho$  corresponds to an automorphism  $\sigma$  of  $\mathbb{Z} \times (\mathbb{Z}K(l))$  acting by increasing the first component by 1. The Serre functor shifted by  $[-1]$  induces an automorphism  $\tau$  of  $\mathbb{Z} \times (\mathbb{Z}K(l))$ , which on vertices maps  $(i, j)$  to  $(i, j - 2)$ .

Taking into account (90), (91) and that the Serre functor maps  $s_i$  to  $s_{i-2}[+1]$  (see the proof of Lemma 7.5) we see that  $\rho$  can be chosen so that  $\rho(i, j) \cong s_j[i]$  for any  $i, j \in \mathbb{Z}$ . Combining [37, Corollary 1.9; Theorem 3.7] and the last paragraph of the proof of [37, Theorem 4.3] we see that  $\text{Aut}(\mathcal{T}_l) \cong \mathbb{Z} \times (\mathbb{Z} \ltimes \text{PGL}_n(k))$  and a generator of the factor  $\mathbb{Z}$  in  $\mathbb{Z} \ltimes \text{PGL}_n(k)$  is the desired  $A_l \in \text{Aut}(\mathcal{T}_l)$ .  $\square$

**7.3. The principal  $\mathbb{C}$ -bundle  $\text{Stab}(\mathcal{T}_l) \rightarrow \mathcal{X}_l$ .** We will denote by  $\mathcal{X}_l$  the set of orbits of the  $\mathbb{C}$ -action on  $\text{Stab}(\mathcal{T}_l)$ , i.e.  $\mathcal{X}_l = \text{Stab}(\mathcal{T}_l)/\mathbb{C}$ . In this section we show that  $\mathcal{X}_l$  is a complex manifold biholomorphic to either  $\mathbb{C}$  or  $\mathcal{H}$  (Corollary 7.10) and that the action of  $\langle A_l \rangle$  on  $\text{Stab}(\mathcal{T}_l)$ , where

$A_l \in \text{Aut}(\mathcal{T}_l)$  is the functor from Corollary 7.6, descends to an action by biholomorphism on  $\mathcal{X}_l$  (Corollary 7.12).

In [35, Section 3.3] are constructed stability conditions generated by a full Ext-exceptional collection. The set of stability conditions generated by a full Ext-exceptional pair  $(A, B)$  will be denoted by  $\Theta'_{(A,B)}$  and for any full exceptional pair<sup>9</sup>  $(A, B)$  the notation  $\Theta_{(A,B)}$  will denote the union of the sets  $\Theta'_{(A[a], B[b])}$  s.t.  $(A[a], B[b])$  is an Ext-pair (see [22, formulas (8), (10)]). The idea of Macrì in [35, Subsection 3.3 and Section 4] how to prove that  $\text{Stab}(\mathcal{T}_l)$  is simply connected is to show that  $\text{Stab}(\mathcal{T}_l)$  is covered by  $\{\Theta_{(s_i, s_{i+1})}\}_{i \in \mathbb{Z}}$  and then use Seifert-Van Kampen theorem. We prove here Proposition 7.7, formula (94), and Lemma 7.8 following this insight of Macrì.

The proof of the following proposition is a simpler analogue of the proof of [22, Proposition 2.7.]:

**Proposition 7.7.** *For  $i \in \mathbb{Z}$  the subset  $\Theta_{(s_i, s_{i+1})} \subset \text{Stab}(\mathcal{T}_l)$  has the following description:*

$$(92) \quad \Theta_{(s_i, s_{i+1})} = \left\{ \sigma \in \text{Stab}(\mathcal{T}) : (s_i, s_{i+1}) \subset \sigma^{ss} \text{ and } \phi_\sigma(s_i) < \phi_\sigma(s_{i+1}) \right\}.$$

In particular, the set  $\Theta_{(s_i, s_{i+1})}$  is biholomorphic to the contractible set  $\mathcal{S} = \{(z_1, z_2) \in \mathbb{C}^2; \Im(z_1) < \Im(z_2)\} \subset \mathbb{C}^2$  by the following map  $\Theta_{(s_i, s_{i+1})} \xrightarrow{\varphi_i} \mathcal{S}$

$$(93) \quad \Theta_{(s_i, s_{i+1})} \ni (Z, \mathcal{P}) \mapsto (\log |Z(s_i)| + i\pi\phi_\sigma(s_i), \log |Z(s_{i+1})| + i\pi\phi_\sigma(s_{i+1})) \in \mathcal{S}.$$

*Proof.* We use [22, formulas (12), (17), (18), and Lemma 2.4] and deduce that  $\sigma \in \Theta_{(s_i, s_{i+1})}$  iff  $(s_i, s_{i+1}) \subset \sigma^{ss}$  and  $(\phi_\sigma(s_i), \phi_\sigma(s_{i+1})) \in \bigcup_{\mathbf{p} \in A_0} S^1(-\infty, 1) - \mathbf{p}$ , where  $S^1(-\infty, 1) = \{(x, y) : x - u < 1\}$  and  $A_0 = \{(0, p) \in \mathbb{Z}^2 : (s_i, s_{i+1}[p]) \text{ is Ext}\}$ . From (90) we see that  $A_0 = \{(0, p) : p \leq -1\}$  and (92) follows. Since the map defined by [22, formulas (9), (11)] is homeomorphism, it follows that (93) is homeomorphism. Let us identify  $\text{Hom}(K_0(\mathcal{T}_l), \mathbb{C}) \cong \mathbb{C}^2$  via the basis  $[s_i], [s_{i+1}]$  of  $K_0(\mathcal{T}_l)$ , and let  $\text{Stab}(\mathcal{T}_l) \xrightarrow{\text{proj}} \text{Hom}(K_0(\mathcal{T}_l), \mathbb{C})$  be the projection  $\text{proj}(Z, \mathcal{P}) = Z$ . Then the following diagram:

$$\begin{array}{ccc} \Theta_{(s_i, s_{i+1})} & \xrightarrow{\text{proj}} & \text{Hom}(K_0(\mathcal{T}_l), \mathbb{C}) \\ \varphi_i \downarrow & & \simeq \downarrow \\ \mathcal{S} & \xrightarrow{\exp \times \exp} & \mathbb{C}^2 \end{array}$$

is commutative. Since the horizontal arrows are local biholomorphisms and we already showed that  $\varphi_i$  is homeomorphism, it follows that  $\varphi_i$  is biholomorphic.  $\square$

[20, Lemma A.1] says that for each  $\sigma \in \text{Stab}(\mathcal{T}_l)$  there exists a  $\sigma$ -exceptional pair. This means that (see [20, Corollary 3.18]) for each  $\sigma$  there exists an Ext-exceptional pair  $(A, B)$  with  $\sigma \in \Theta'_{(A,B)}$ . From (84) we see that  $(A, B)$  is of the form  $(s_i[a], s_{i+1}[b])$  for some  $i, a, b \in \mathbb{Z}$ . Therefore:

$$(94) \quad \text{Stab}(\mathcal{T}_l) = \bigcup_{i \in \mathbb{Z}} \Theta_{(s_i, s_{i+1})}.$$

**Lemma 7.8.** *Let  $i, j$  be two different integers. Then the equality below holds*

$$(95) \quad \Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})} = \left\{ \sigma \in \text{Stab}(\mathcal{T}_l) : (s_i, s_{i+1}) \subset \sigma^{ss} \text{ and } \phi_\sigma(s_i) < \phi_\sigma(s_{i+1}) < \phi_\sigma(s_i) + 1 \right\},$$

and therefore  $\Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})} = \bigcap_{p \in \mathbb{Z}} \Theta_{(s_p, s_{p+1})}$ . It follows that  $\text{Stab}(\mathcal{T}_l)$  is contractible.

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<sup>9</sup>not necessarily Ext

*Proof.* We show first the inclusion  $\supset$ . So let  $(s_i, s_{i+1}) \subset \sigma^{ss}$  and  $\phi_\sigma(s_i) < \phi_\sigma(s_{i+1}) < \phi_\sigma(s_i) + 1$ . We will show below that the given inequalities imply:

$$(96) \quad \sigma \in \Theta_{(s_{i-1}, s_i)} \cap \Theta_{(s_{i+1}, s_{i+2})}; \quad \begin{array}{l} \phi_\sigma(s_{i-1}) < \phi_\sigma(s_i) < \phi_\sigma(s_{i-1}) + 1 \\ \phi_\sigma(s_{i+1}) < \phi_\sigma(s_{i+2}) < \phi_\sigma(s_{i+1}) + 1 \end{array}.$$

Then by induction we obtain the inclusion  $\subset$ . We use [22, Proposition 2.2] first to show that  $s_{i-1}$  and  $s_{i+2}$  are semi-stable. More precisely, the given inequality is the same as  $\phi_\sigma(s_{i+1}[-1]) < \phi_\sigma(s_i) < \phi_\sigma(s_{i+1})$ , which together with (90) imply that  $s_i, s_{i+1}[-1]$  is a  $\sigma$ -exceptional pair (see [20, Definition 3.17]). From two consecutive triangles in the sequence of triangles (86) it follows that  $s_{i-1}$  and  $s_{i+2}[-1]$  are in the extension closure of  $s_i, s_{i+1}[-1]$ . Since we have also  $\text{hom}(s_i, s_{i+1}) \neq 0$ , we can apply [22, Proposition 2.2] and it ensures that  $s_{i-1}, s_{i+2}[-1] \in \sigma^{ss}$ . Now from (90), (91) it follows:

$$(97) \quad \phi(s_{i-1}) \leq \phi(s_i) \leq \phi(s_{i+1}) \leq \phi(s_{i+2}) \leq \phi(s_{i-1}) + 1 \leq \phi(s_{i+1}) + 1.$$

The given  $\phi_\sigma(s_i) < \phi_\sigma(s_{i+1}) < \phi_\sigma(s_i) + 1$  amounts to  $\frac{Z(s_{i+1})}{Z(s_i)} \in \mathcal{H}$  and then Lemma 7.4 ensures that  $\frac{Z(s_i)}{Z(s_{i-1})}, \frac{Z(s_{i+2})}{Z(s_{i+1})} \in \mathcal{H}$ , which together with (97) leads to  $\phi(s_{i-1}) < \phi(s_i) < \phi(s_{i-1}) + 1$ ,  $\phi(s_{i+1}) < \phi(s_{i+2}) < \phi(s_{i+1}) + 1$ . Thus we derived (96) and the inclusion  $\supset$  follows.

To show the opposite inclusion  $\subset$  it is enough to consider the case  $i < j$ . So if  $\sigma \in \Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})}$ , then (92) shows that  $s_i, s_{i+1}, s_j, s_{j+1} \in \sigma^{ss}$  and  $\phi(s_i) < \phi(s_{i+1})$ ,  $\phi(s_j) < \phi(s_{j+1})$ , therefore using (90), (91) we get:

$$(98) \quad \phi(s_i) < \phi(s_{i+1}) \leq \phi(s_j) < \phi(s_{j+1}) \leq \phi(s_i) + 1$$

and the equality (95) is proved. The map  $\varphi_i$  in (93) restricts to a biholomorphism

$$\Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})} \xrightarrow{\varphi_i} \{(z_1, z_2) \in \mathbb{C}^2; \Im(z_1) < \Im(z_2) < \Im(z_1) + 1\} \subset \mathbb{C}^2$$

hence  $\Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})} = \bigcap_{p \in \mathbb{Z}} \Theta_{(s_p, s_{p+1})}$  is contractible. Since  $\text{Stab}(\mathcal{T}_l)$  is covered by the contractible sets  $\{\Theta_{(s_i, s_{i+1})}\}_{i \in \mathbb{Z}}$  (see (94)), using [22, Remark A.6] we deduce that  $\text{Stab}(\mathcal{T}_l)$  is contractible.  $\square$

**Definition 7.9.** We will denote the quotient  $\text{Stab}(\mathcal{T}_l)/\mathbb{C}$  by  $\mathcal{X}_l$ , the corresponding projection by  $\text{Stab}(\mathcal{T}_l) \xrightarrow{\text{pr}} \mathcal{X}_l$ . The intersection  $\bigcap_{p \in \mathbb{Z}} \Theta_{(s_p, s_{p+1})}$  will be denoted by  $\mathcal{Z}$ . Due to Lemma 7.8 we have  $\mathcal{Z} = \Theta_{(s_i, s_{i+1})} \cap \Theta_{(s_j, s_{j+1})}$  for any  $i \neq j$ .

From (94) we get a disjoint union  $\text{Stab}(\mathcal{T}_l) = \mathcal{Z} \amalg \amalg_{i \in \mathbb{Z}} (\Theta_{(s_i, s_{i+1})} \setminus \mathcal{Z})$ .

Corollary 3.3 and Lemma 7.8 imply:

**Corollary 7.10.**  $\mathcal{X}_l$  is biholomorphic either to  $\mathbb{C}$  or to  $\mathcal{H}$  and  $\text{Stab}(\mathcal{T}_l) \xrightarrow{\text{pr}} \mathcal{X}_l$  is trivial  $\mathbb{C}$ -principal bundle.

The action (39) descends to an action by biholomorphisms on  $\mathcal{X}_l$ . To show this and some basic properties of this action we note first:

**Lemma 7.11.** For any  $i, j \in \mathbb{Z}$ , any  $\lambda \in \mathbb{C}$ , and  $A_l \in \text{Aut}(\mathcal{T}_l)$  from Corollary 7.6 hold the equalities:

$$(99) \quad \lambda \star \Theta_{(s_i, s_{i+1})} = \Theta_{(s_i, s_{i+1})} \quad A_l^j \cdot \Theta_{(s_i, s_{i+1})} = \Theta_{(s_{i+j}, s_{i+j+1})} \quad \lambda \star \mathcal{Z} = \mathcal{Z} \quad A_l^j \cdot \mathcal{Z} = \mathcal{Z}.$$

*Proof.* From (32) we see that the conditions  $s_i, s_{i+1} \in \sigma^{ss}$ ,  $\phi_\sigma(s_i) < \phi_\sigma(s_{i+1})$  are equivalent to the conditions:  $s_i, s_{i+1} \in (z \star \sigma)^{ss}$ ,  $\phi_{z \star \sigma}(s_i) < \phi_{z \star \sigma}(s_{i+1})$ , hence by (92) we obtain the first equality.

From Lemma 7.6 by induction we get  $A_l^j(s_i) \cong s_{i+j}$ ,  $A_l^j(s_{i+1}) \cong s_{i+j+1}$ . Now with the help of (40) and (92) we establish the second equality by a sequence of equivalences:

$$\begin{aligned} \sigma \in \Theta_{(s_i, s_{i+1})} &\iff (s_i, s_{i+1}) \subset \sigma^{ss} \iff (A_l^j(s_i), A_l^j(s_{i+1})) \subset \overline{A_l^j(\sigma^{ss})} = (A_l^j \cdot \sigma)^{ss} \\ &\iff \phi_\sigma(s_i) < \phi_\sigma(s_{i+1}) \iff \phi_\sigma((A_l^j)^{-1}(A_l^j(s_i))) < \phi_\sigma((A_l^j)^{-1}(A_l^j(s_{i+1}))) \\ &\iff (s_{i+j}, s_{i+j+1}) \subset (A_l^j \cdot \sigma)^{ss} \iff A_l^j \cdot \sigma \in \Theta_{(s_{i+j}, s_{i+j+1})}. \end{aligned}$$

The third and the fourth equalities in (99) follow from the already proven first and second.  $\square$

**Corollary 7.12.** *The action (39) descends to an action by biholomorphisms on  $\mathcal{X}_l$  by the formula  $\Phi \cdot \text{pr}(\sigma) = \text{pr}(\Phi \cdot \sigma)$ . Denote  $\Theta_i = \text{pr}(\Theta_{(s_i, s_{i+1})})$  for  $i \in \mathbb{Z}$  and  $\mathfrak{Z} = \text{pr}(\mathcal{Z})$ . Then for any  $i, j \in \mathbb{Z}$ ,  $j \neq 0$  we have: (a)  $\Theta_{(s_i, s_{i+1})} = \text{pr}^{-1}(\Theta_i)$ ; (b)  $\mathcal{Z} = \text{pr}^{-1}(\mathfrak{Z})$ ; (c)  $\{\Theta_i\}_{i \in \mathbb{Z}}$  is an open cover of  $\mathcal{X}_l$ ;*

$$(100) \quad (d) \quad \mathfrak{Z} = \bigcap_{p \in \mathbb{Z}} \Theta_p = \Theta_i \cap \Theta_{i+j} \quad A_l^j \cdot \Theta_i = \Theta_{i+j} \quad A_l^j \cdot \mathfrak{Z} = \mathfrak{Z}.$$

*Proof.* Since the actions (31), (39) commute and  $\text{pr}$  from Corollary 7.10 has holomorphic sections, it follows the first sentence. The rest of the corollary follows from the definition of quotient topology on  $\mathcal{X}_l$  and from Lemmas 7.11, 7.8.  $\square$

**7.4. The action  $\langle A_l \rangle$  on  $\mathcal{X}_l$  is free and properly discontinuous for  $l \geq 2$ .** In this Section we complete the proof of Theorem 1.1.

Corollary 7.12 gives a decomposition of  $\mathcal{X}_l$  into  $\langle A_l \rangle$ -invariant subsets  $\mathfrak{Z}$  and  $\mathcal{X}_l \setminus \mathfrak{Z}$  and furthermore it gives a decomposition  $\mathcal{X}_l \setminus \mathfrak{Z} = \coprod_{i \in \mathbb{Z}} (\Theta_i \setminus \mathfrak{Z})$  into subsets which  $\langle A_l \rangle$  permutes, more precisely:

$$(101) \quad \langle A_l \rangle \cdot \mathfrak{Z} = \mathfrak{Z} \quad A_l^j \cdot (\Theta_i \setminus \mathfrak{Z}) = \Theta_{i+j} \setminus \mathfrak{Z} \quad \mathcal{X}_l \setminus \mathfrak{Z} = \coprod_{i \in \mathbb{Z}} (\Theta_i \setminus \mathfrak{Z}).$$

Now we construct biholomorphisms between  $\mathcal{H}$ ,  $\Theta_j$  and  $\mathfrak{Z}$  and describe the action of  $\langle A_l \rangle$  on  $\mathfrak{Z}$ .

**Lemma 7.13.** *Let  $j \in \mathbb{Z}$ . Let  $\mathcal{H} \xrightarrow{\gamma} \mathcal{S}$  be the map  $\gamma(z) = (0, z)$  and let  $\varphi_j$  be as in (93). Then the function (102) is a biholomorphism.*

$$(102) \quad \mathcal{H} \xrightarrow{\gamma} \mathcal{S} \xrightarrow{\varphi_j^{-1}} \Theta_{(s_j, s_{j+1})} \xrightarrow{\text{pr}_l} \Theta_j.$$

*This function, restricted to the strip  $\{z \in \mathbb{C} : 0 < \Im(z) < \pi\}$ , is a biholomorphism between this strip and  $\mathfrak{Z} = \text{pr}(\mathcal{Z})$ . In particular, we obtain a biholomorphism:*

$$(103) \quad \mathcal{H} \xrightarrow{\psi} \mathfrak{Z} \quad \psi(z) = \text{pr} \circ \varphi_j^{-1} \circ \gamma(\log |z| + i \text{Arg}(z)).$$

*We claim that for any  $p \in \mathbb{Z}$  and any  $z \in \mathcal{H}$  we have:*

$$(104) \quad A_l^{-p} \cdot \psi(z) = \psi(\alpha_l^p(z)),$$

*where  $\alpha_l$  is the matrix in Lemma 7.4 and  $\alpha_l^p(z)$  is defined in (79).*

*Proof.* Since all composing maps in (102) are holomorphic, the composite function is also holomorphic. We will construct an inverse holomorphic function. Let  $\mathcal{S} \xrightarrow{pr_i} \mathbb{C}$  be the projections  $pr_i(z_1, z_2) = z_i$ ,  $i = 1, 2$  and let  $\kappa_1, \kappa_2$  be the holomorphic functions (105), which obviously satisfy (106):

$$(105) \quad \mathbb{C} \xleftarrow{\kappa_1} \Theta_{(s_j, s_{j+1})} \xrightarrow{\kappa_2} \mathbb{C} \quad \kappa_1 = pr_1 \circ \varphi_j \quad \kappa_2 = pr_2 \circ \varphi_j$$

$$(106) \quad \kappa_2 \circ \varphi_j^{-1} \circ \gamma = Id_{\mathcal{H}} \quad \kappa_1 \circ \varphi_j^{-1} \circ \gamma = 0.$$

One computes by straightforward application of (32), (33), and (93):

$$(107) \quad \kappa_1(\lambda \star \sigma) = \lambda + \kappa_1(\sigma) \quad \kappa_2(\lambda \star \sigma) = \lambda + \kappa_2(\sigma) \quad \text{for } \lambda \in \mathbb{C}, \sigma \in \Theta_{(s_j, s_{j+1})}$$

therefore  $\kappa_1((- \kappa_1(\sigma)) \star \sigma) = 0$  and  $\varphi_j((- \kappa_1(\sigma)) \star \sigma) = (0, \kappa_2((- \kappa_1(\sigma)) \star \sigma)) \in \mathcal{S}$ , hence  $\kappa_2((- \kappa_1(\sigma)) \star \sigma) \in \mathcal{H}$  and we can define a holomorphic function (108) satisfying (109), (110) (recall also (106)):

$$(108) \quad \Theta_{(s_j, s_{j+1})} \xrightarrow{\kappa} \mathcal{H} \quad \kappa(\sigma) = \kappa_2((- \kappa_1(\sigma)) \star \sigma) = \kappa_2(\sigma) - \kappa_1(\sigma) \quad \text{for } \sigma \in \Theta_{(s_j, s_{j+1})}$$

$$(109) \quad \varphi_j^{-1} \circ \gamma \circ \kappa(\sigma) = (- \kappa_1(\sigma)) \star \sigma \quad \kappa(\lambda \star \sigma) = \kappa(\sigma) \quad \text{for } \lambda \in \mathbb{C}, \sigma \in \Theta_{(s_j, s_{j+1})}$$

$$(110) \quad \kappa \circ \varphi_j^{-1} \circ \gamma = Id_{\mathcal{H}}.$$

Due to the equality  $\kappa(\lambda \star \sigma) = \kappa(\sigma)$  we get a well (and uniquely) defined function

$$\Theta_j \xrightarrow{\kappa'} \mathcal{H} \quad \kappa' \circ \mathbf{pr}_j = \kappa.$$

Furthermore, since  $\mathbf{pr}_j$  is a holomorphic  $\mathbb{C}$ -principal bundle (Corollaries 7.12, 7.10), it has holomorphic sections and it follows that  $\kappa'$  is holomorphic. We claim that  $\kappa'$  is the desired inverse of (102).

Indeed, using (110) we obtain  $\kappa' \circ (\mathbf{pr}_j \circ \varphi_j^{-1} \circ \gamma) = \kappa \circ \varphi_j^{-1} \circ \gamma = Id_{\mathcal{H}}$ . Using (109) we compute  $(\mathbf{pr}_j \circ \varphi_j^{-1} \circ \gamma) \circ \kappa' \circ \mathbf{pr}_j = \mathbf{pr}_j$ , which implies  $(\mathbf{pr}_j \circ \varphi_j^{-1} \circ \gamma) \circ \kappa' = Id_{\Theta_j}$ , since  $\mathbf{pr}_j$  is surjective.

To consider the restriction of (102) to the strip we take  $z \in \mathbb{C}$  with  $0 < \Im(z) < \pi$  and let  $\sigma = \varphi_j^{-1} \circ \gamma(z)$ , then  $\varphi_j(\sigma) = (0, z)$  and (93) shows that  $s_j, s_{j+1} \in \sigma^{ss}$ ,  $\phi_\sigma(s_j) = 0 < \Im(z)/\pi = \phi_\sigma(s_{j+1}) < 1$ , therefore  $\sigma \in \mathcal{Z}$  (see (95)), and hence  $\mathbf{pr}(\sigma) \in \mathfrak{Z}$ . Conversely, take any  $\mathbf{pr}(\sigma) \in \mathfrak{Z}$ ,  $\sigma \in \mathcal{Z}$ , then  $\phi_\sigma(s_j) < \phi_\sigma(s_{j+1}) < \phi_\sigma(s_j) + 1$  and by (107), (108) we get  $\kappa'(\mathbf{pr}(\sigma)) = \kappa(\sigma) = -\kappa_1(\sigma) + \kappa_2(\sigma)$ , therefore (93):  $\Im(\kappa'(\mathbf{pr}(\sigma))) = \Im(-\kappa_1(\sigma) + \kappa_2(\sigma)) = \pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)) < \pi$ . Thus we get (103).

Let  $\sigma \in \mathcal{Z}$ , recalling that  $\kappa'$  is the inverse of (102) and the definition of  $\psi$  in (103), we compute

$$\begin{aligned} \psi^{-1}(\mathbf{pr}(\sigma)) &= \exp \circ \kappa'(\mathbf{pr}(\sigma)) = \exp \circ \kappa(\sigma) = \exp(\kappa_2(\sigma) - \kappa_1(\sigma)) \\ &= \text{by (93)} = \exp(\log |Z_\sigma(s_{j+1})| + i\pi\phi_\sigma(s_{j+1}) - \log |Z_\sigma(s_j)| - i\pi\phi_\sigma(s_j)) = \\ &= \text{by (19)} = \frac{Z_\sigma(s_{j+1})}{Z_\sigma(s_j)}. \end{aligned}$$

To show (104) we first recall that for  $\sigma \in \mathcal{Z}$  we have  $\{s_i\}_{i \in \mathbb{Z}} \subset \sigma^{ss}$ ,  $\left\{ \frac{Z_\sigma(s_{i+1})}{Z_\sigma(s_i)} \right\}_{i \in \mathbb{Z}} \subset \mathcal{H}$  (Lemma 7.8) and  $A_l^p \cdot \sigma \in \mathcal{Z}$  for any  $p \in \mathbb{Z}$  (see (99)), then using what we just computed, we get (here

$\sigma \in \mathcal{Z}$ ):

$$\begin{aligned} \psi^{-1}(A_l^{-p} \cdot \mathbf{pr}(\sigma)) &= \psi^{-1}(\mathbf{pr}(A_l^{-p} \cdot \sigma)) = \frac{Z_{A_l^{-p} \cdot \sigma}(s_{j+1})}{Z_{A_l^{-p} \cdot \sigma}(s_j)} = \text{by (41)} = \frac{Z_\sigma(A_l^p(s_{j+1}))}{Z_\sigma(A_l^p(s_j))} \\ &= \text{by Corollary 7.6} = \frac{Z_\sigma(s_{p+j+1})}{Z_\sigma(s_{p+j})} = \text{by Lemma 7.4} = \alpha_l^p \left( \frac{Z_\sigma(s_{j+1})}{Z_\sigma(s_j)} \right) = \alpha_l^p(\psi^{-1}(\mathbf{pr}(\sigma))). \end{aligned}$$

Now (104) follows from the obtained  $\psi^{-1}(A_l^{-p} \cdot \mathbf{pr}(\sigma)) = \alpha_l^p(\psi^{-1}(\mathbf{pr}(\sigma)))$ , since  $\psi$  is biholomorphism.  $\square$

The matrix  $\alpha_l = \begin{bmatrix} l & -1 \\ 1 & 0 \end{bmatrix}$  has  $\text{tr}(\alpha_l) = l$ . Since  $l \geq 2$ , then  $\alpha_l$  is either parabolic or hyperbolic (see Section 7.1.1). As a corollary following from (101), Remark 7.2 (a) and (104) we get:

**Corollary 7.14.** *The action of  $\langle A_l \rangle$  on  $\mathcal{X}_l$  is free and its restriction to  $\mathfrak{Z}$  is properly discontinuous.*

The next step is to find a fundamental domain of the action of  $\langle A_l \rangle$  on  $\mathfrak{Z} \cong \mathcal{H}$ . We choose  $j \in \mathbb{Z}$  and use the biholomorphism  $\psi : \mathcal{H} \rightarrow \mathfrak{Z}$  (103) (depending on  $j$ ) constructed in Lemma 7.13. By (104) we see that the fundamental domain we want is of the form  $\psi(F_l)$ , where  $F_l \subset \mathcal{H}$  is a fundamental domain of the action of  $\langle \alpha_l \rangle$  on  $\mathcal{H}$  discussed in Section 7.1. We use now Sections 7.1.2 and 7.1.3 to find such a  $F_l$ .

**Lemma 7.15.** *Let us denote  $a_l = \frac{l+\sqrt{l^2-4}}{2}$  for  $l \geq 2$ ,  $a_2 = 1$ . A fundamental domain of  $\langle \alpha_l \rangle$  is:*

$$\begin{aligned} (111) \quad F_l &= \left\{ x + iy \in \mathcal{H} : \frac{a_l^2+1}{2a_l} (x^2 + y^2) \geq x \leq \frac{a_l^2+1}{2a_l} \right\} \\ \text{Bd}_{\mathcal{H}}(F_l) &= \left\{ x + iy \in \mathcal{H} : \frac{a_l^2+1}{2a_l} (x^2 + y^2) = x \text{ or } x = \frac{a_l^2+1}{2a_l} \right\}. \end{aligned}$$

For  $l \geq 3$  the fundamental domain is shown in Figure (2a) and for  $l = 2$  in Figure (2b).

Two points in  $F_l$  lie in a common orbit iff they satisfy  $z_{\pm} \in \text{Bd}_{\mathcal{H}}(F_l)$ ,  $\text{Arg}(z_+) = \text{Arg}(z_-)$ .

For each  $u \in \text{Bd}_{\mathcal{H}}(F_l)$  there exists an open subset  $U \subset \mathcal{H}$ , s. t.  $u \in U$  and  $\{i \in \mathbb{Z} : \alpha^i(U) \cap F_l \neq \emptyset\}$  is finite (in fact contains only two elements).

*Proof.* We need to consider two cases  $l \geq 3$  and  $l = 2$ , because this determines the type of  $\alpha_l$  according to the classification recalled in Subsection 7.1.1.

Assume first that  $l \geq 3$ . The feature of  $a_l$  we need, which we have for all  $l \geq 3$ , is:  $a_l > 1$ , that's why we will omit the subscript  $l$  and write just  $a$ , remembering that  $a > 1$ . Now  $\alpha_l$  is a hyperbolic element in  $\text{SL}(2, \mathbb{Z})$  (see Subsection 7.1.1) and we can use the method described in Subsection 7.1.2.

We note first that  $\beta^{-1} \alpha_l \beta = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ , where

$$(112) \quad a = \frac{l + \sqrt{l^2 - 4}}{2} \quad \beta^{-1} = \frac{1}{a - a^{-1}} \begin{bmatrix} 1 & -a^{-1} \\ -1 & a \end{bmatrix} \quad \beta = \begin{bmatrix} a & a^{-1} \\ 1 & 1 \end{bmatrix}.$$

Since  $a > 1$ , it follows that  $\det(\beta) > 0$  and  $\beta$  determines a biholomorphism of  $\mathcal{H}$  (recall (79)):

$$(113) \quad \beta(z) = \frac{a^2 z + 1}{a(z + 1)} \quad z \in \mathcal{H}.$$

We choose the parameters of the strip  $F'$  in Figure (1a) to be  $\delta = 1/a^2$ ,  $\delta a^2 = 1$ . Since  $\beta(1) = \frac{a^2+1}{2a}$ ,  $\beta(-1) = \infty$ , by Remark 7.1 we see that  $\beta(\mathbf{b}'_+) = \mathbf{b}_+$  (see Figures (1a) and (2a)). Since  $\beta(\frac{-1}{a^2}) = 0$ ,  $\beta(\frac{1}{a^2}) = \frac{2a}{a^2+1}$  and using Remark 7.1 again, we deduce that  $\beta(\mathbf{b}'_-) = \mathbf{b}_-$ . Finally, since  $\beta(1/a) = 1$  we deduce that  $\beta(F') = F_l$ , where  $F'$ ,  $F_l$  are depicted in Figures (1a) and (2a). The first part of the lemma is proved for  $l \geq 3$ . Two points in  $F_l$  lie in a common orbit iff they are of the form  $\beta(a^2z), \beta(z), z \in \mathbf{b}'_-$ . For  $z \in \mathbf{b}'_-$  we have  $|z|^2 = 1/a^4$  and we compute:

$$\frac{\beta(a^2z)}{\beta(z)} = \frac{(a^4z+1)(z+1)}{(a^2z+1)^2} = \frac{(a^4z+1)(z+1)}{(a^2z+1)^2} \frac{\bar{z}^2}{\bar{z}^2} = \frac{(1+\bar{z})(\frac{1}{a^4}+\bar{z})}{(\frac{1}{a^2}+\bar{z})^2} = \frac{(1+\bar{z})(1+a^4\bar{z})}{(1+a^2\bar{z})^2} = \frac{\overline{\beta(a^2z)}}{\overline{\beta(z)}}$$

hence  $\beta(a^2z), \beta(z)$  are parallel for  $z \in \mathbf{b}'_-$  and, being in a common quadrant,  $\beta(a^2z) \in \mathbb{R}^{>0} \beta(z)$ .

Assume that  $l = 2$ . In this case  $\alpha_2$  is parabolic and we can use the method described in Subsection 7.1.3. We note first that  $\beta^{-1} \alpha_2 \beta = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ , where  $\beta^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix}$   $\beta = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ . And now  $\beta(F')$  is a fundamental domain, where  $F'$  is of the form in Figure (1b) and  $\beta$  is:

$$(114) \quad \beta(z) = \frac{2z-1}{2z+1} \quad z \in \mathcal{H}.$$

We choose the parameters of the strip  $F'$  in Figure (1a) to be  $\delta = -1/2$ ,  $\delta + |b/a| = 1/2$ . Since  $\beta(1/2) = 0$ ,  $\beta(\infty) = 1$ ,  $\beta(-1/2) = \infty$ , by Remark 7.1 we see that  $\beta(\mathbf{b}'_{\pm}) = \mathbf{b}_{\pm}$  (see Figures (1b) and (2b)). Finally, since  $\beta(0) = -1$ , we deduce that  $\beta(F') = F_2$ , where  $F'$ ,  $F_2$  are depicted in Figures (1b) and (2b). It remains to describe the orbits in  $F_2$ . Two points in  $F_2$  lie in a common orbit iff they are of the form  $\beta(z-1), \beta(z), z \in \mathbf{b}'_+$ . For  $z \in \mathbf{b}'_+$  we have  $\Re(z) = 1/2$  and with the computation below the lemma is proved (recall also Remark 7.2 (b)):

$$\frac{\beta(z)}{\beta(z-1)} = \frac{(2z-1)^2}{(2z+1)(2z-3)} = \frac{(i2\Im(z))^2}{(2+2i\Im(z))(2i\Im(z)-2)} = \frac{\Im(z)^2}{1+\Im(z)^2} > 0.$$

□

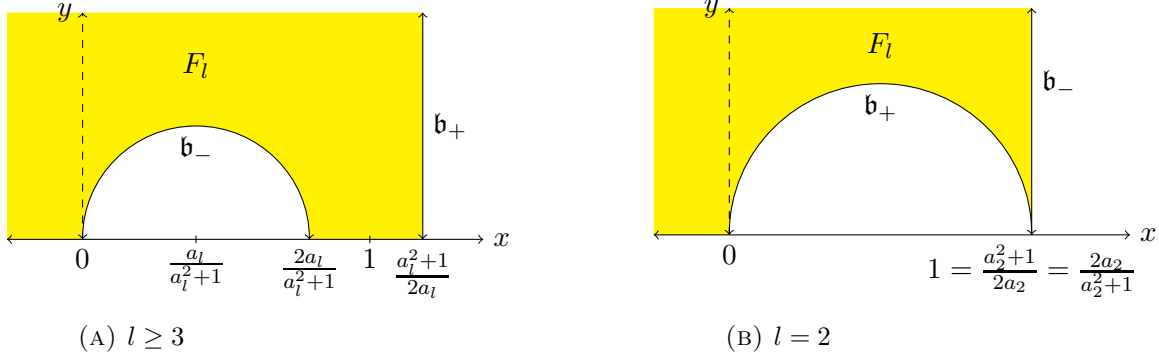
**Corollary 7.16.** *A fundamental domain  $\mathcal{F}'_l$  of the action  $\langle A_l \rangle$  on  $\mathfrak{Z}$  is the image of the set:*

$$(115) \quad \begin{aligned} & \{(u+iv) \in \mathbb{C} : 0 < v < \pi \text{ and } e^{u+\Delta_l} \geq \cos(v) \leq e^{-u+\Delta_l}\} \\ & = \{(u+iv) \in \mathbb{C} : v \in (0, \pi) \text{ and } v \geq \arccos(\min\{1, e^{\Delta_l-|u|}\})\} \end{aligned}$$

by the biholomorphism (102), where  $\Delta_l = \log\left(\frac{a_l^2+1}{2a_l}\right)$ . The following specifications hold:

- (a) The boundary  $\text{Bd}_3(\mathcal{F}'_l)$  is the image of  $\{\pm u + i \arccos(e^{\Delta_l-|u|}) : |u| > \Delta_l\}$  by (102).
- (b) Two points in  $\mathcal{F}'_l$  lie in the same orbit of the action iff they are in  $\text{Bd}_3(\mathcal{F}'_l)$  and have the same imaginary parts via the inverse of (102).
- (c) For any  $q \in \text{Bd}_3(\mathcal{F}'_l)$  there exists an open  $U \subset \mathfrak{Z}$ , s. t.  $q \in U$  and  $\{i \in \mathbb{Z} : A^i(U) \cap \mathcal{F}'_l \neq \emptyset\}$  is finite (in fact contains only two elements).

*Proof.* Recalling (102), (103) we see that  $\mathcal{F}'_l = \psi(F_l) = \text{pr}_l \circ \varphi_j^{-1} \circ \gamma \left( \exp_{\{z \in \mathcal{H} : \Im(z) < \pi\}}^{-1}(F_l) \right)$ . So we have to determine  $\exp_{\{z \in \mathcal{H} : \Im(z) < \pi\}}^{-1}(F_l)$ . An element  $(u+iv) \in \mathbb{C}$  is in the latter set iff  $0 < v < \pi$  and  $\exp(u+iv) \in F_l$ , which by Lemma 7.15 is the same as:  $\frac{a_l^2+1}{2a_l} e^{2u} \geq \cos(v) e^u \leq \frac{a_l^2+1}{2a_l}$ ,

FIGURE 2. Fundamental domains of  $\langle \alpha_l \rangle$ 

so we get

$$(116) \quad \mathcal{F}'_l = \text{pr}_1 \circ \varphi_j^{-1} \circ \gamma \left( \left\{ (u + iv) \in \mathcal{H} : v < \pi \quad \frac{a_l^2 + 1}{2a_l} e^u \geq \cos(v) \leq e^{-u} \frac{a_l^2 + 1}{2a_l} \right\} \right)$$

and the corollary follows from (104), the fact that  $\psi$  is biholomorphism, and the properties of  $F_l$  obtained in Lemma 7.15.  $\square$

**Corollary 7.17.** *Let  $j \in \mathbb{Z}$ . Let  $\Theta_j \xrightarrow{u+iv} \mathcal{H}$  be the inverse of (102). Let  $\Delta_l = \log \left( \frac{a_l^2 + 1}{2a_l} \right)$ . The subset  $\mathcal{F}_l = \{q \in \Theta_j : v(q) \geq \arccos(\min\{1, e^{\Delta_l - |u(q)|}\})\}$  has the following properties:*

- (a)  $\text{Bd}_3(\mathcal{F}'_l) = \text{Bd}_{\Theta_j}(\mathcal{F}_l) = \{q \in \Theta_j : v(q) = \arccos(\min\{1, e^{\Delta_l - |u(q)|}\})\} \subset \mathfrak{Z}$ .
- (b) *The interior of  $\mathcal{F}_l$  w.r. to  $\mathcal{X}_l$  is  $\mathcal{F}_l^o = \{q \in \Theta_j : v(q) > \arccos(\min\{1, e^{\Delta_l - |u(q)|}\})\}$ .*
- (c)  $\mathcal{X}_l = \cup_{i \in \mathbb{Z}} A_l^i(\mathcal{F}_l)$
- (d)  $A_l^m(\mathcal{F}_l^o) \cap A_l^n(\mathcal{F}_l^o) = \emptyset$  for  $m \neq n$ .
- (e) *Two points in  $\mathcal{F}_l$  lie in the same orbit of the  $\langle A_l \rangle$ -action on  $\mathcal{X}_l$  iff they are in  $\text{Bd}_{\Theta_j}(\mathcal{F}_l)$  and have the same imaginary parts (i.e.  $v$  has the same values on them).*
- (f) *For each  $q \in \text{Bd}_{\Theta_j}(\mathcal{F}_l)$  there exists an open subset  $U \subset \mathcal{X}_l$ , s. t.  $q \in U$  and  $\{i \in \mathbb{Z} : A^i(U) \cap \mathcal{F}_l \neq \emptyset\}$  is finite (in fact contains only two elements).*

*Proof.* (a): It is clear that  $\text{Bd}_{\Theta_j}(\mathcal{F}_l) = \{q \in \Theta_j : v(q) = \arccos(\min\{1, e^{\Delta_l - |u(q)|}\})\}$ , since (102) is biholomorphism.  $\text{Bd}_3(\mathcal{F}'_l)$  is the same set by Corollary 7.16 (a).

(b): Since  $\Theta_j$  is open subset in  $\mathcal{X}_l$ , the interiors of  $\mathcal{F}_l$  w.r. to  $\Theta_j$  and w.r. to  $\mathcal{F}_l$  coincide. Hence (b) is due to the fact that (102) is biholomorphism.

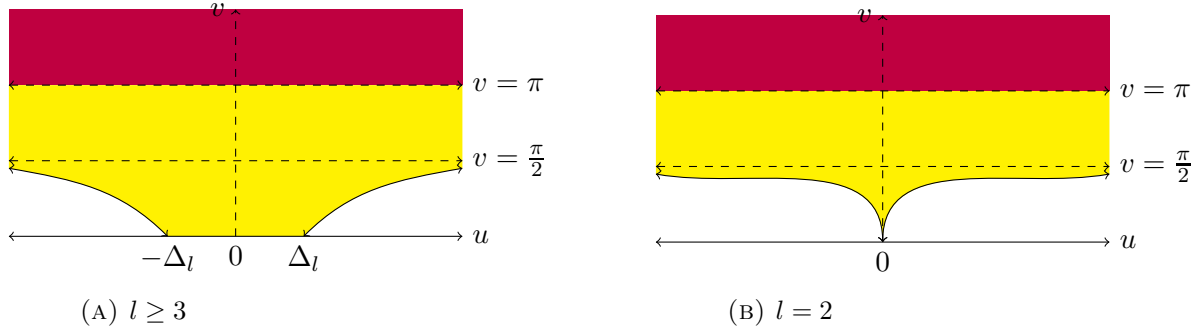
(c) and (d): From Lemma 7.13 and the definition of  $\mathcal{F}'_l$  in Corollary 7.16 we see that

$$(117) \quad \mathcal{F}_l = \mathcal{F}'_l \cup (\Theta_j \setminus \mathfrak{Z}) \quad \mathcal{F}_l^o = \mathcal{F}'_l^o \cup (\Theta_j \setminus \mathfrak{Z}).$$

Therefore by (101) and since  $\mathcal{F}'_l$  is fundamental domain of the action of  $\langle A_l \rangle$  on  $\mathfrak{Z}$  (as defined on [38, p. 20]) we get:  $\cup_{i \in \mathbb{Z}} A_l^i(\mathcal{F}_l) = \cup_{i \in \mathbb{Z}} A_l^i(\mathcal{F}'_l) \cup \cup_{i \in \mathbb{Z}} A_l^i(\Theta_j \setminus \mathfrak{Z}) = \mathfrak{Z} \cup (\mathcal{X}_l \setminus \mathfrak{Z}) = \mathcal{X}_l$  and  $A_l^m(\mathcal{F}_l^o) \cap A_l^n(\mathcal{F}_l^o) = (A_l^m(\mathcal{F}_l^o) \cup A_l^m(\Theta_j \setminus \mathfrak{Z})) \cap (A_l^n(\mathcal{F}_l^o) \cup A_l^n(\Theta_j \setminus \mathfrak{Z})) = \emptyset$  for  $m \neq n$ .

(e): From (117) and (101) we see that if two points in  $\mathcal{F}_l$  lie in a common orbit, then they lie in  $\mathcal{F}'_l$ , and then we use the already proven (a) here and Corollary 7.16 (b) to obtain (e).




 FIGURE 3.  $(u + iv)(\mathcal{F}_l) = \mathfrak{F}_l$ 

(f): For  $q \in \text{Bd}_{\Theta_j}(\mathcal{F}_l)$  the same neighbourhood  $U \ni q$  ensured by Corollary 7.16 (c) satisfies the required property in (f) due to (101), (117), and the already proven (a) here.  $\square$

**Corollary 7.18.** *The action of  $\langle A_l \rangle$  on  $\mathcal{X}_l$  is free and properly discontinuous.*

*Proof.* In Corollary 7.14 we already showed that the action is free. Take any two  $q_1, q_2 \in \mathcal{X}_l$ . We need to find open subsets  $U_i \ni q_i$  in  $\mathcal{X}_l$ ,  $i = 1, 2$ , such that the set  $\{i \in \mathbb{Z} : A_l^i(U_1) \cap U_2 \neq \emptyset\}$  is finite (see e.g. [38, p. 17]). From (c) in Corollary 7.17 it follows that it is enough to consider the case  $q_1, q_2 \in \mathcal{F}_l$ .<sup>10</sup> If  $q_1, q_2 \in \text{Bd}_{\Theta_j}(\mathcal{F}_l)$ , then  $q_1, q_2 \in \mathfrak{Z}$  (see Corollary 7.17 (a)) and the neighborhoods  $U_i \ni q_i$  we need exist since  $\mathfrak{Z}$  is an  $\langle A_l \rangle$ -invariant open subset and by Corollary 7.14. If  $q_1, q_2 \in \mathcal{F}_l^\circ$ , then  $U_1 = U_2 = F_l^\circ$  satisfy the condition we need by Corollary 7.17 (d). Since  $\mathcal{F}_l = F_l^\circ \cup \text{Bd}_{\Theta_j}(\mathcal{F}_l)$ , it remains to consider the case  $q_1 \in \text{Bd}_{\Theta_j}(\mathcal{F}_l)$ ,  $q_2 \in F_l^\circ$ . In this case we take  $U \ni q_1$  as in Corollary 7.17 (f) and put  $U_1 = U$ ,  $U_2 = \mathcal{F}_l^\circ$ , and the corollary is proved.  $\square$

**Corollary 7.19.** *The orbit-space  $\mathcal{X}_l / \langle A_l \rangle$  with the quotient topology carries a structure of a one dimensional complex manifold, s. t. the projection  $\tilde{\text{pr}} : \mathcal{X}_l \rightarrow \mathcal{X}_l / \langle A_l \rangle$  is a holomorphic covering map: a universal cover of  $\mathcal{X}_l / \langle A_l \rangle$ . In particular  $\pi_1(\mathcal{X}_l / \langle A_l \rangle) = \mathbb{Z}$ .*

*Proof.* In the previous corollary we showed that the action is free and proper. From these properties [31, Proposition 1.2] ensures that  $\mathcal{X}_l / \langle A_l \rangle$  has the structure of a one dimensional complex manifold, s. t.  $\tilde{\text{pr}} : \mathcal{X}_l \rightarrow \mathcal{X}_l / \langle A_l \rangle$  is a holomorphic principal  $\mathbb{Z}$ -bundle and recalling that  $\mathcal{X}_l$  is contractible (see Corollary 7.10) the corollary follows from the long exact sequence for the homotopy groups associated to  $\tilde{\text{pr}}$ .  $\square$

From now on we fix  $j \in \mathbb{Z}$  and let  $\mathcal{F}_l \subset \Theta_j$  be the closed in  $\Theta_j$  subset obtained in Corollary 7.17. By (c) in Corollary 7.17 it follows that  $\tilde{\text{pr}}(\mathcal{F}_l) = \tilde{\text{pr}}(\Theta_j) = \mathcal{X}_l / \langle A_l \rangle$ , i. e. we have the surjectivity of the restriction of the map  $\tilde{\text{pr}}$  from Corollary 7.19:

**Lemma 7.20.** *The restriction  $\tilde{\text{pr}}|_{\mathcal{F}_l} : \mathcal{F}_l \rightarrow \mathcal{X}_l / \langle A_l \rangle$  is a proper surjective map.*

*Proof.* It remains to show the properness. Recall that  $\Theta_j$  is an open subset in  $\mathcal{X}_l$ , and  $\mathcal{F}_l$  is a closed subset in  $\Theta_j$ . Thus, the restriction  $\Theta_j \xrightarrow{\tilde{\text{pr}}|_{\Theta_j}} \mathcal{X}_l / \langle A_l \rangle$  is a local biholomorphism between locally compact spaces. Let  $K \subset \mathcal{X}_l / \langle A_l \rangle$  be compact.

<sup>10</sup>because the set  $\{i \in \mathbb{Z} : A_l^i(A_l^m U_1) \cap A_l^n U_2\}$  is  $n - m + \{i \in \mathbb{Z} : A_l^i(U_1) \cap U_2\}$

For any  $q \in K$  we fix  $q' \in \mathcal{F}_l$ , s. t.  $\tilde{\mathbf{pr}}(q') = q$ . Now we will choose an open  $U_q \ni q'$  subset of  $\Theta_j$  with certain properties for each  $q \in K$ . If  $q' \in \mathcal{F}_l^o$  we choose  $U_q \ni q'$  whose closure w. r. to  $\Theta_j$  is compact and contained in  $\mathcal{F}_l^o$  (in particular  $A_l^j(U_q) \cap \mathcal{F}_l = \emptyset$  for  $j \neq 0$ ). If  $q' \in \text{Bd}_{\Theta_j}(\mathcal{F}_l)$  we choose  $U_q \ni q'$  with compact closure w. r. to  $\Theta_j$ , which is contained in  $\mathfrak{Z}$  and such that  $\{i \in \mathbb{Z} : A_l^i(U_q) \cap \mathcal{F}_l \neq \emptyset\}$  is finite (we can do this by Corollary 7.17 (f) and since  $\text{Bd}_{\Theta_j}(\mathcal{F}_l) \subset \mathfrak{Z}$ ). Since  $K$  is compact and  $\tilde{\mathbf{pr}}$  is open, we have  $K \subset \cup_{i=1}^n \tilde{\mathbf{pr}}(U_{q_i})$  for a finite family  $\{q_i\}_{i=1}^n \subset K$ , i.e.

$$(\tilde{\mathbf{pr}}|_{\mathcal{F}_l})^{-1}(K) \subset \cup_{i=1}^n \tilde{\mathbf{pr}}^{-1}(\tilde{\mathbf{pr}}(U_{q_i})) \cap \mathcal{F}_l = \cup_{i=1}^n \cup_{m \in \mathbb{Z}} A_l^m(U_{q_i}) \cap \mathcal{F}_l.$$

By our choice of  $U_q$  the union on the right hand side is finite and each element  $A_l^m(U_{q_i})$  in it is contained in a compact subset of  $\Theta_j$  (recall also (101)). Since  $\mathcal{F}_l$  is closed in  $\Theta_j$ , we deduce that  $(\tilde{\mathbf{pr}}|_{\mathcal{F}_l})^{-1}(K)$  is contained in a compact subset of  $\mathcal{F}_l$ , and therefore, being closed subset of compact,  $(\tilde{\mathbf{pr}}|_{\mathcal{F}_l})^{-1}(K)$  is compact.  $\square$

Now we can prove:

**Proposition 7.21.** *If  $l \geq 3$ , then  $\mathcal{X}_l$  is biholomorphic to  $\mathcal{H}$ , and Corollary 7.10 implies Theorem 1.1.*

*Proof.* Suppose that  $\mathcal{X}_l$  is not biholomorphic to  $\mathcal{H}$ . We will obtain a contradiction.

By Corollary 7.10 we see that we have a biholomorphism  $\mathcal{X}_l \cong \mathbb{C}$ , and we showed in Corollary 7.19 that  $\tilde{\mathbf{pr}} : \mathcal{X}_l \rightarrow \mathcal{X}_l / \langle A_l \rangle$  is a universal covering of  $\mathcal{X}_l / \langle A_l \rangle$  and  $\pi_1(\mathcal{X}_l / \langle A_l \rangle) = \mathbb{Z}$ . [25, Prop. 27.12] ensures that  $\mathcal{X}_l / \langle A_l \rangle$  is biholomorphic to one of these:  $\mathbb{C}, \mathbb{C}^*$ , or a torus. However  $\pi_1(\mathcal{X}_l / \langle A_l \rangle) = \mathbb{Z}$  and we deduce that  $\theta : \mathcal{X}_l / \langle A_l \rangle \cong \mathbb{C}^*$  for some biholomorphism  $\theta$ . Let us denote by  $f$  the following composition:

$$(118) \quad f : \mathcal{H} \xrightarrow{\gamma} \mathcal{S} \xrightarrow{\varphi_j^{-1}} \Theta_{(s_j, s_{j+1})} \xrightarrow{\mathbf{pr}_l} \Theta_j \xrightarrow{\tilde{\mathbf{pr}}_l} \mathcal{X}_l / \langle A_l \rangle \xrightarrow{\theta} \mathbb{C}^*.$$

As in Corollary 7.17 we denote by  $\Theta_j \xrightarrow{u+iv} \mathcal{H}$  the inverse of (102). Let us denote  $\mathfrak{F}_l = (u+iv)(\mathcal{F}_l)$  (see Figure (3a)). From Corollary 7.17 (or Corollary 7.16) (a) we see that:

$$(119) \quad \text{Bd}_{\mathcal{H}}(\mathfrak{F}_l) = \{\pm\delta(t) + it : t \in (0, \pi/2)\}, \quad \text{where} \quad \delta(t) = (\Delta_l - \log(\cos(t))).$$

Then from Lemma 7.13, Corollary 7.17, and Lemma 7.20 follow (a), (b), (c) below:

- (a)  $f : \mathcal{H} \rightarrow \mathbb{C}^*$  is a local biholomorphism;
- (b)  $f(\mathfrak{F}_l) = \mathbb{C}^*$  and  $\mathfrak{F}_l \xrightarrow{f|_{\mathcal{F}_l}} \mathbb{C}^*$  is proper;
- (c)  $q_{\pm} \in \mathfrak{F}_l$  and  $f(q_+) = f(q_-)$  implies  $q_+ = q_-$  or  $q_{\pm} = \pm\delta(t) + it$  for some  $t \in (0, \pi/2)$ .

For any  $t \in (0, \pi/2)$  let us denote the segment  $\gamma_t$  in  $\mathcal{F}_l$  defined by  $\gamma_t(s) = s + it$  for  $s \in [-\delta(t), \delta(t)]$ . Then (see Figure (4))

$$(120) \quad \mathfrak{F}_l = \mathfrak{F}_l^{t-} \amalg \text{im}(\gamma_t) \amalg \mathfrak{F}_l^{t+}, \quad \text{where} \quad \mathfrak{F}_l^{t-} = \{z \in \mathfrak{F}_l : \Im(z) < t\}, \quad \mathfrak{F}_l^{t+} = \{z \in \mathfrak{F}_l : \Im(z) > t\}.$$

From (c) above it follows that  $f(\mathcal{F}_l^{t+})$ ,  $f(\text{im}(\gamma_t))$ ,  $f(\mathcal{F}_l^{t-})$  are pairwise disjoint and we can write :

$$(121) \quad \mathbb{C}^* = f(\mathfrak{F}_l^{t+}) \amalg f(\text{im}(\gamma_t)) \amalg f(\mathfrak{F}_l^{t-}) \quad f|_{\mathfrak{F}_l}^{-1}(f(\mathfrak{F}_l^{t\pm})) = \mathfrak{F}_l^{t\pm} \quad f|_{\mathfrak{F}_l}^{-1}(f(\text{im}(\gamma_t))) = \text{im}(\gamma_t).$$

Due to (c) above  $f \circ \gamma_t$  is a closed Jordan curve in  $\mathbb{C}^*$ . By Jordan curve theorem (see e.g. [34, Theorem 4.13 on p. 701]) we have  $\mathbb{C} \setminus f(\text{im}(\gamma_t)) = U_t \amalg V_t$ , where  $U_t, V_t$  are the connected components of the complement of  $\text{im}(f \circ \gamma_t)$  in  $\mathbb{C}$ . In particular, since  $f(\mathfrak{F}_l^{t\pm})$  are connected, by (121) we get either  $f(\mathfrak{F}_l^{t+}) \subset U_t \setminus \{0\}$ ,  $f(\mathfrak{F}_l^{t-}) \subset V_t \setminus \{0\}$  or  $f(\mathfrak{F}_l^{t-}) \subset U_t \setminus \{0\}$ ,  $f(\mathfrak{F}_l^{t+}) \subset V_t \setminus \{0\}$ .

Due to the first equality in (121) we can write:  $f(\mathfrak{F}_l^{t-}) \amalg f(\mathfrak{F}_l^{t+}) = (U_t \setminus \{0\}) \amalg (V_t \setminus \{0\})$  and we obtain:

$$(122) \quad \{f(\mathfrak{F}_l^{t+}) = U_t \setminus \{0\} \text{ and } f(\mathfrak{F}_l^{t-}) = V_t \setminus \{0\}\} \text{ or } \{f(\mathfrak{F}_l^{t-}) = U_t \setminus \{0\} \text{ and } f(\mathfrak{F}_l^{t+}) = V_t \setminus \{0\}\}.$$

Furthermore, by Jordan theorem we can assume that  $U_t \cup f(\text{im}(\gamma_t))$  is compact in  $\mathbb{C}$ ,  $V_t \cup \text{im}(\gamma_t)$  is closed, non-compact in  $\mathbb{C}$ . We claim that  $0 \in U_t$ . Indeed, if  $0 \notin U_t$ , then  $U_t \setminus \{0\} = U_t$  and by (122) either  $f(\mathfrak{F}_l^{t+} \cup \text{im}(\gamma_t))$  or  $f(\mathfrak{F}_l^{t-} \cup \text{im}(\gamma_t))$  is compact, which in turn by (121) and (b) above leads to either  $\mathfrak{F}_l^{t+} \cup \text{im}(\gamma_t)$  or  $\mathfrak{F}_l^{t-} \cup \text{im}(\gamma_t)$  is compact, which is a contradiction (see (120)). So we see that  $0 \in U_t$ ,  $V_t = V_t \setminus \{0\}$ , and  $\{f(\mathfrak{F}_l^{t+}) = U_t \setminus \{0\} \text{ and } f(\mathfrak{F}_l^{t-}) = V_t\}$  or  $\{f(\mathfrak{F}_l^{t-}) = U_t \setminus \{0\} \text{ and } f(\mathfrak{F}_l^{t+}) = V_t\}$  for any  $t \in (0, \pi/2)$ . We will show that we can ensure:

$$(123) \quad \forall t \in (0, \pi/2) \quad 0 \in U_t \text{ and } f(\mathfrak{F}_l^{t-}) = U_t \setminus \{0\} \text{ and } f(\mathfrak{F}_l^{t+}) = V_t.$$

Indeed, if for some  $t \in (0, \pi/2)$  this holds, then  $f(\mathfrak{F}_l^{t'-}) = U_{t'} \setminus \{0\}$ ,  $f(\mathfrak{F}_l^{t'+}) = V_{t'}$  for any other  $t' \in (0, \pi/2)$ .<sup>11</sup> On the other hand by composing  $\theta$  in (118) with  $\frac{1}{z}$  we can ensure that the equalities in (123) hold for some  $t$  and then they will hold for any  $t \in (0, \pi/2)$ . Next step is to show:

$$(124) \quad \{z_i\}_{i \in \mathbb{N}} \subset \mathfrak{F}_l \text{ and } \lim_{i \rightarrow \infty} \Im(z_i) = 0 \quad \Rightarrow \quad \lim_{i \rightarrow \infty} |f(z_i)| = 0.$$

We can assume that  $\{z_i\}_{i \in \mathbb{N}} \subset \mathfrak{F}_l^{t-}$  for some  $t \in (0, \pi/2)$  and therefore by (123)  $\{f(z_i)\}_{i \in \mathbb{N}} \subset U_t \setminus \{0\}$  is a bounded sequence. If  $\lim_{i \rightarrow \infty} |f(z_i)| = 0$  does not hold, then for some subsequence  $\{z_{i_m}\}_{m \in \mathbb{N}}$  holds  $\lim_{i \rightarrow \infty} f(z_{i_m}) = q \in \mathbb{C}^*$ . Now  $\{f(z_{i_m})\}_{m \in \mathbb{N}} \cup \{q\}$  is a compact subset of  $\mathbb{C}^*$ , but  $f_{\mathfrak{F}_l}^{-1}(\{f(z_{i_m})\}_{m \in \mathbb{N}} \cup \{q\}) \supset \{z_{i_m}\}_{m \in \mathbb{N}}$  is not compact, since  $\lim_{i \rightarrow \infty} \Im(z_{i_m}) = 0$  and  $f_{\mathfrak{F}_l}^{-1}(\{f(z_{i_m})\}_{m \in \mathbb{N}} \cup \{q\})$  does not contain any point with zero imaginary part, so we get a contradiction to (b) and proved (124).

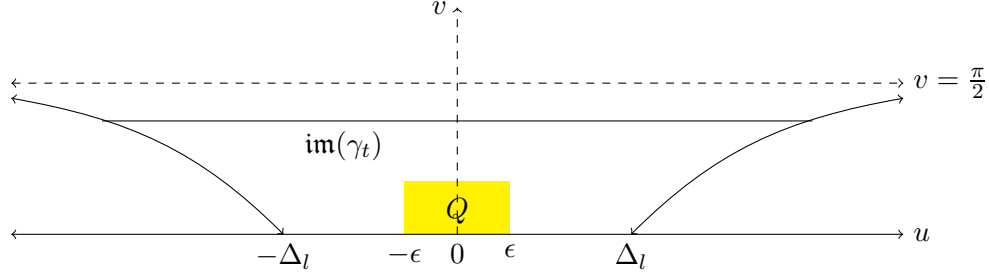
Since  $l \geq 3$ , then  $\Delta_l = \log\left(\frac{a_l^2 + 1}{2a_l}\right) > 0$  (recall that  $a_l = \frac{l + \sqrt{l^2 - 4}}{2}$ ). Therefore we can choose  $0 < \epsilon < \min\{\Delta_l, \pi/4\}$  and denote (see figure (4)):

$$(125) \quad Q = \{u + iv \in \mathcal{H} : \max\{u, v\} < \epsilon\}.$$

Since  $Q \subset \mathfrak{F}_l^o$  is an open subset and  $Q \subset \mathfrak{F}_l^{\frac{\pi}{4}-}$ , by (a), (c) and (123) it follows that  $Q \xrightarrow{f|_Q} f(Q)$  is a biholomorphism between open simply-connected bounded subsets in  $\mathbb{C}$ . Obviously  $\text{Bd}_{\mathbb{C}}(Q) = \text{Bd}_{\mathcal{H}}(Q) \amalg [-\epsilon, \epsilon]$  is (image of) a closed Jordan curve. On the other hand from (124) we see that, if exists, any continuous extension of  $f|_Q$  to  $\text{Cl}_{\mathbb{C}}(Q)$  must map the entire real segment  $[-\epsilon, \epsilon]$  to  $\{0\}$  and thus it cannot be one-to-one. Therefore by boundary correspondence theorem [34, Theorem 2.24 on p. 70<sup>3</sup>]  $\text{Bd}_{\mathbb{C}}(f(Q))$  cannot be (image of) a closed Jordan curve, otherwise a one-to-one extension of  $f|_Q$  to a homeomorphism between  $\text{Cl}_{\mathbb{C}}(Q)$  and  $\text{Cl}_{\mathbb{C}}(f(Q))$  must exist. We will prove the theorem by deriving the contrary:  $\text{Bd}_{\mathbb{C}}(f(Q))$  is an image of a closed Jordan curve.

Since  $f|_{\mathfrak{F}_l}$  is proper, it follows that it is a closed map (see e.g. [25, p. 35]), therefore  $f(Q \amalg \text{Bd}_{\mathcal{H}}(Q))$  is closed in  $\mathbb{C}^*$ , and it follows that  $f(Q) \cup f(\text{Bd}_{\mathcal{H}}(Q)) \cup \{0\}$  is closed in  $\mathbb{C}$ . Hence  $\text{Cl}_{\mathbb{C}}(f(Q)) \subset f(Q) \cup f(\text{Bd}_{\mathcal{H}}(Q)) \cup \{0\}$ . On the other hand (124) shows that  $\{0\} \in \text{Cl}_{\mathbb{C}}(f(Q))$  and

<sup>11</sup>Otherwise we would have  $f(\mathfrak{F}_l^{t'-}) = V_{t'}$ ,  $f(\mathfrak{F}_l^{t'+}) = U_{t'} \setminus \{0\}$ , which implies  $V_{t'} \subset U_t$  for  $t' < t$ , which is impossible or  $U_t \setminus \{0\} \subset V_{t'}$  for  $t < t'$ . The latter in turn implies the contradiction  $U_{t'} \cap V_{t'} \neq \emptyset$ , since  $(U_t \cap U_{t'}) \setminus \{0\} \neq \emptyset$ .

FIGURE 4.  $l \geq 3$ 

$f(\text{Bd}_{\mathcal{H}}(Q)) \subset \text{Cl}_{\mathbb{C}}(f(Q))$  by continuity, hence  $\text{Cl}_{\mathbb{C}}(f(Q)) = f(Q) \amalg f(\text{Bd}_{\mathcal{H}}(Q)) \amalg \{0\}$ . Since  $f(Q)$  is open in  $\mathbb{C}$ , we obtain:

$$(126) \quad \text{Bd}_{\mathbb{C}}(f(Q)) = f(\text{Bd}_{\mathcal{H}}(Q)) \amalg \{0\}.$$

Take any homeomorphism  $(0, 2\pi) \xrightarrow{\kappa} \text{Bd}_{\mathcal{H}}(Q)$ , s. t.  $\lim_{t \rightarrow 0} \kappa(t) = -\epsilon$ ,  $\lim_{t \rightarrow 2\pi} \kappa(t) = +\epsilon$ , then  $(0, 2\pi) \xrightarrow{f \circ \kappa} f(\text{Bd}_{\mathcal{H}}(Q))$  is a homeomorphism and by (124) we have  $\lim_{t \rightarrow 0} f(\kappa(t)) = \lim_{t \rightarrow 2\pi} f(\kappa(t)) = 0$ , then it is clear that the following function (127) is continuous and bijective, hence by compactness of  $\mathbb{S}^1$  it is homeomorphism and the theorem is proved.

$$(127) \quad \mathbb{S}^1 \xrightarrow{\zeta} f(\text{Bd}_{\mathcal{H}}(Q)) \amalg \{0\} \quad \zeta(\exp(it)) = \begin{cases} f(\kappa(t)) & t \in (0, 2\pi) \\ 0 & t = 0 \end{cases}$$

□

**7.5. The set of phases.** Let  $l \geq 2$  be an integer. Recall that we denote

$$(128) \quad a_l = \frac{l + \sqrt{l^2 - 4}}{2} \Rightarrow a_l^{-1} = \frac{l - \sqrt{l^2 - 4}}{2}; \quad a_l^{-1} + a_l = \frac{a_l^2 + 1}{a_l} = l.$$

In this section we write for short  $P_{\sigma}^l$  instead of  $P_{\sigma}^{\mathcal{T}_l}$ , and will determine  $P_{\sigma}^l$  (Proposition 7.23).

We start by some comments on the root system of  $K(l)$ . The root system of  $K(l)$  is  $\Delta_{l+} = \Delta_{l+}^{re} \cup \Delta_{l+}^{im}$ , where  $\Delta_{l+}^{re} = \{(n, m) \in \mathbb{N}^2 : n^2 + m^2 - lmn = 1\}$  and  $\Delta_{l+}^{im} = \{(n, m) \in \mathbb{N}^2 : n^2 + m^2 - lmn \leq 0\} \setminus \{(0, 0)\}$ . It is well known that the real roots  $\Delta_{l+}^{re}$  are exactly the dimension vectors of the exceptional representations in  $\text{Rep}_k(K(l))$  and for the imaginary roots  $\Delta_{l+}^{im}$  we have formula (74) in [19]. From (84) and Lemma 7.5 we have the complete list  $\{s_{\leq 0}[1], s_{\geq 1}\}$  of exceptional representations in  $\text{Rep}_k(K(l))$ . Let us denote the corresponding dimension vectors as follows:

$$(129) \quad (n_i, m_i) = \begin{cases} \underline{\dim}(s_i) & i \geq 1 \\ \underline{\dim}(s_i[1]) & i \leq 0 \end{cases}$$

Therefore we can write:

$$(130) \quad \Delta_{l+}^{re} = \{(n_i, m_i) : i \in \mathbb{Z}\} \quad \Delta_{l+} = \{(n_i, m_i) : i \in \mathbb{Z}\} \cup \left\{ a_l^{-1} \leq \frac{n}{m} \leq a_l : n \in \mathbb{N}_{\geq 1}, m \in \mathbb{N}_{\geq 1} \right\}.$$

We will need later the following facts for the real roots  $\{(n_i, m_i) : i \in \mathbb{Z}\}$ :

- Lemma 7.22.** (a)  $(n_{-1}, m_{-1}) = (l, 1)$ ,  $(n_0, m_0) = (1, 0)$ ,  $(n_1, m_1) = (0, 1)$ ,  $(n_2, m_2) = (1, l)$   
 (b)  $(m_{-i}, n_{-i}) = (n_{i+1}, m_{i+1})$  for  $i \geq 0$ .  
 (c)  $n_{-i} > m_{-i}$  and  $n_{i+1} < m_{i+1}$  for  $i \geq 0$ ;  $n_{i+1} > 0$  and  $m_{-i} > 0$  for  $i \geq 1$ .  
 (d)  $\frac{n_i}{m_i} = \frac{l}{2} - \sqrt{\frac{l^2}{4} - 1 + \frac{1}{m_i^2}}$  and  $\frac{n_{-i}}{m_{-i}} = \frac{l}{2} + \sqrt{\frac{l^2}{4} - 1 + \frac{1}{m_{-i}^2}}$  for  $i \geq 1$ .  
 (e)  $\frac{n_{-1}}{m_{-1}} > \frac{n_{-2}}{m_{-2}} > \dots > \frac{n_{-i}}{m_{-i}} \xrightarrow{i \rightarrow \infty} a_l$  and  $0 = \frac{n_1}{m_1} < \frac{n_2}{m_2} < \dots < \frac{n_i}{m_i} \xrightarrow{i \rightarrow \infty} a_l^{-1}$ .

*Proof.* (a)  $(n_0, m_0) = (1, 0)$ ,  $(n_1, m_1) = (0, 1)$  follow from the definition. The triangles (86) for  $i = 1$ , and  $i = 0$  amount to short exact sequences:  $s_1^l \rightarrow s_2 \rightarrow s_0[1]$  and  $s_1 \rightarrow s_{-1}[1] \rightarrow s_0[1]^l$  in  $\text{Rep}_k(K(l))$ , and it follows that  $\underline{\dim}(s_2) = (1, l)$ ,  $\underline{\dim}(s_{-1}[1]) = (l, 1)$

(b) The equality for  $0 \leq i \leq 1$  follows from (a). We make the induction assumption that for some  $p \geq 1$  the equality holds for any  $0 \leq i \leq p$ , we will make the induction step, namely that the equality for  $i = p + 1$  follows from this induction assumption. Indeed, for  $i \geq 1$  from (86) we obtain the following short exact sequences in  $\text{Rep}_k(K(l))$ :  $s_{-i-1}[1] \rightarrow s_{-i}^l[1] \rightarrow s_{-i+1}[1]$ ,  $s_i \rightarrow s_{i+1}^l \rightarrow s_{i+2}$  therefore for  $i \geq 1$  we obtain:

$$(131) \quad n_{-i-1} = l n_{-i} - n_{-i+1} \quad m_{-i-1} = l m_{-i} - m_{-i+1}$$

$$(132) \quad n_{i+2} = l n_{i+1} - n_i \quad m_{i+2} = l m_{i+1} - m_i$$

having these recursive formulas one easily carries out the inductive step.

(c) Due to (b) it is enough to show that  $n_{i+1} < m_{i+1}$ . For  $i = 0$  this is shown in (a). For  $i \geq 1$  we have  $\text{hom}(s_{i+1}, s_{-i}[1]) > 0$ ,  $\text{hom}^1(s_{i+1}, s_{-i}[1]) = 0$  (recall Lemma 7.5), hence the Euler formula amounts to:

$$\begin{aligned} \langle \underline{\dim}(s_{i+1}), \underline{\dim}(s_{-i}[1]) \rangle &= \text{hom}(s_{i+1}, s_{-i}[1]) - \text{hom}^1(s_{i+1}, s_{-i}[1]) > 0 \\ \Rightarrow \langle (n_{i+1}, m_{i+1}), (n_{-i}, m_{-i}) \rangle &= n_{i+1} n_{-i} + m_{i+1} m_{-i} - l n_{i+1} m_{-i} > 0. \end{aligned}$$

Putting the equality from (b) in the last inequality we get  $n_{i+1} m_{i+1} + m_{i+1} n_{i+1} - l n_{i+1} n_{i+1} > 0$ . Therefore  $n_{i+1}(2m_{i+1} - l n_{i+1}) > 0$ , hence  $n_{i+1} > 0$ ,  $m_{i+1} > \frac{l}{2} n_{i+1} \geq n_{i+1}$  and (c) is proved.

(d) Take any  $i \in \mathbb{Z}$ ,  $i \neq 0$ . From (c) we know that  $m_i \neq 0$ . From (130) we know that  $n_i^2 + m_i^2 - l n_i m_i = 1$ , hence via the quadratic equation  $\left(\frac{n_i}{m_i}\right)^2 - l \frac{n_i}{m_i} + 1 - \frac{1}{m_i^2} = 0$  we get  $\frac{n_i}{m_i} = \frac{1}{2} \left( l \pm \sqrt{l^2 - 4 + \frac{4}{m_i^2}} \right)$ . One checks that  $\frac{1}{2} \left( l + \sqrt{l^2 - 4 + \frac{4}{m_i^2}} \right) > 1$ ,  $\frac{1}{2} \left( l - \sqrt{l^2 - 4 + \frac{4}{m_i^2}} \right) < 1$  and then from (c) we deduce (d).

(e) Using (131), (132), and induction one shows that  $m_i < m_{i+1}$ ,  $m_{-i-1} > m_{-i}$  for  $i \geq 0$  and then (e) follows from (d).  $\square$

Now we have the necessary notations to describe  $P_\sigma^l$ :

**Proposition 7.23.** Let  $\sigma \in \text{Stab}(D^b(K(l)))$ .

- (a) If  $\sigma \notin \mathcal{Z}$  (see also Definition 7.9), then the set of phases  $P_\sigma^l$  is finite (has up to 4 elements).  
 (b) If  $\sigma \in \mathcal{Z}$ , then for any  $j \in \mathbb{Z}$  we have the following formulas:

$$(133) \quad 0 < \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) < 1$$

$$(134) \quad \exp(i\pi(1 - \phi_\sigma(s_j))) \cdot P_\sigma^l = \{\pm 1\} \cup \{\pm \exp(i\pi(n_i/m_i)) : i \neq 0\} \cup \pm \exp(i\pi(\mathbb{Q} \cap [a_l^{-1}, a_l])),$$

where  $x$ ,  $y$ , and the function (strictly increasing smooth)  $f : [0, \infty) \rightarrow [\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)), \pi)$  are:

$$(135) \quad f(t) = \arccos \left( \frac{xy - t}{\sqrt{t^2 + x^2 - 2txy}} \right) \quad x = \frac{|Z(s_{j+1})|}{|Z(s_j)|} \quad y = \cos(\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j))).$$

(c) For  $\sigma \in \mathcal{Z}$  holds the equality  $\{\pm \exp(i\pi\phi_\sigma(s_i))\}_{i \in \mathbb{Z}} = P_\sigma^l \setminus L(P_\sigma^l)$  (recall that by  $L(P_\sigma^l)$  we denote the set of limit points in the circle of  $P_\sigma^l$ ).

(d) For any  $\sigma \in \mathcal{Z}$  and any  $j \in \mathbb{Z}$  hold:

$$(136) \quad \lim_{k \rightarrow +\infty} \pi\phi_\sigma(s_k[-1]) = u_\sigma \leq v_\sigma = \lim_{k \rightarrow -\infty} \pi\phi_\sigma(s_k)$$

$$(137) \quad \begin{aligned} \pi(\phi_\sigma(s_j) - 1) &< \pi\phi_\sigma(s_{j+1}[-1]) < \pi\phi_\sigma(s_{j+2}[-1]) < \dots < u_\sigma \leq \\ &\leq v_\sigma < \dots < \pi\phi_\sigma(s_{j-2}) < \pi\phi_\sigma(s_{j-1}) < \pi\phi_\sigma(s_j) \end{aligned}$$

$$(138) \quad \overline{P_\sigma^l} = \pm \exp(i\{\pi\phi_\sigma(s_{j+k}[-1])\}_{k \geq 1} \cup i[u_\sigma, v_\sigma] \cup i\{\pi\phi_\sigma(s_{j-k})\}_{k \geq 0})$$

$$(139) \quad \frac{v_\sigma - u_\sigma}{u_\sigma - \pi\phi_\sigma(s_{j+1}[-1])} = \frac{f(a_l) - f(a_l^{-1})}{f(a_l^{-1}) - \arccos(y)} \quad \frac{v_\sigma - u_\sigma}{\pi\phi_\sigma(s_j) - v_\sigma} = \frac{f(a_l) - f(a_l^{-1})}{\pi - f(a_l)},$$

where  $f$ ,  $x$ ,  $y$  are as in (135) and  $u_\sigma = f(a_l^{-1}) + \pi(1 - \phi_\sigma(s_j))$ ,  $v_\sigma = f(a_l) + \pi(1 - \phi_\sigma(s_j))$ .

(e) Let  $\sigma \in \mathcal{Z}$  and  $0 < \varepsilon < 1$ . Then  $\mathbb{S}^1 \setminus P_\sigma^l$  contains a closed  $\varepsilon$ -arc iff there exists  $i \in \mathbb{Z}$  such that  $\phi_\sigma(s_{i+1}) - \phi_\sigma(s_i) > \varepsilon$ .

Before giving the proof of this proposition we make some preparatory steps.

For a pair of complex numbers  $v = (z_1, z_2)$  we discussed in [19] (see [19, Lemma 3.18] and the first row of the proof) the following subset of the circle

$$(140) \quad R_{v, \Delta_{l+}} = \left\{ \pm \frac{nz_1 + mz_2}{|nz_1 + mz_2|} \mid (n, m) \in \Delta_{l+} \right\} \subset \mathbb{S}^1.$$

From [19, Remark 3.16] and (130) we deduce that:

**Lemma 7.24.** *For any pair of complex numbers  $v = (z_1, z_2)$  of the form  $z_i = r_i \exp(i\phi_i)$ ,  $r_i > 0$ ,  $i = 1, 2$ ,  $0 < \phi_2 < \phi_1 \leq \pi$  holds:*

$$(141) \quad R_{v, \Delta_{l+}} = \{\pm \exp(i\phi_1)\} \cup \{\pm \exp(i f(n_i/m_i)) : i \neq 0\} \cup \{\pm \exp(i f(n/m)) : n/m \in [a_l^{-1}, a_l]\}$$

where  $f : [0, \infty) \rightarrow [\phi_2, \phi_1) \subset (0, \pi)$  is the strictly increasing smooth function:

$$(142) \quad f(t) = \arccos \left( \frac{tr_1 \cos(\phi_1) + r_2 \cos(\phi_2)}{\sqrt{t^2 r_1^2 + r_2^2 + 2tr_1 r_2 \cos(\phi_1 - \phi_2)}} \right), \quad f(0) = \phi_2, \quad \lim_{t \rightarrow \infty} f(t) = \phi_1.$$

*Proof of Proposition 7.23.* Let  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(D^b(K(l)))$ . From Lemma 7.8 and equality (94) we have either  $\sigma \in \Theta_{(s_j, s_{j+1})} \setminus \mathcal{Z}$  for some  $j \in \mathbb{Z}$  or  $\sigma \in \mathcal{Z}$  (see also Definition 7.9).

(a) Assume first that  $\sigma \in \Theta_{(s_j, s_{j+1})} \setminus \mathcal{Z}$  for some  $j \in \mathbb{Z}$ . Then by Lemmas 7.7 and 7.8 we see that

$$(143) \quad s_j, s_{j+1} \in \sigma^{ss} \quad \phi(s_j) + 1 \leq \phi(s_{j+1}).$$

We will show that in this case  $P_\sigma^l = \{\pm \exp(i\pi\phi_\sigma(s_j)), \pm \exp(i\pi\phi_\sigma(s_{j+1}))\}$ . Indeed, (143) implies that there exists  $k \geq 1$  such that  $\phi(s_j) \leq \phi(s_{j+1}[-k]) < \phi(s_j) + 1$ . From Lemma 7.5 it follows that  $(s_j, s_{j+1}[-k])$  is a  $\sigma$ -exceptional pair (as defined in [20, Definition 3.17]). From [20, Corollary

3.18] (and its proof) it follows that the extension closure of  $(s_j, s_{j+1}[-k])$  equals  $\mathcal{P}(t, t+1]$  for some  $t \in \mathbb{R}$ . Since  $(s_j, s_{j+1}[-k])$  is an exceptional pair, each element  $Y$  in the extension closure of  $(s_j, s_{j+1}[-k])$  can be put in a triangle of the form  $s_{j+1}[-k]^a \longrightarrow Y \longrightarrow s_j^b \longrightarrow s_{j+1}[-k+1]^b$  for some  $a, b \in \mathbb{N}$ . Take any  $X \in \sigma^{ss}$ , then for some  $i \in \mathbb{Z}$  we have  $\phi_\sigma(X[i]) \in (t, t+1]$  and therefore we have a triangle:

$$(144) \quad s_{j+1}[-k]^a \xrightarrow{\alpha} X[i] \xrightarrow{\beta} s_j^b \longrightarrow s_{j+1}[-k+1]^a.$$

If  $a = 0$  or  $b = 0$ , then  $X[i] \cong s_j^b$  or  $X[i] \cong s_{j+1}[-k]^a$  and hence  $\phi_\sigma(X[i]) = \phi(s_j)$  or  $\phi_\sigma(X[i]) = \phi(s_{j+1}[-k])$  and the  $\exp(i\pi\phi_\sigma(X)) \in \{\pm \exp(i\pi\phi_\sigma(s_j)), \pm \exp(i\pi\phi_\sigma(s_{j+1}))\}$ .

Next assume that  $a \neq 0$  and  $b \neq 0$ . If  $\phi(s_j) = \phi(s_{j+1}[-k])$ , then we obtain  $\exp(i\pi\phi_\sigma(X)) = \exp(i\pi\phi_\sigma(s_j))$  using (144). Thus, we reduce to  $\phi(s_j) < \phi(s_{j+1}[-k])$ , which in turn by (144),  $X[i] \in \sigma^{ss}$ , and (20) implies that  $\alpha = 0$  or  $\beta = 0$ . If  $\alpha = 0$ , then  $s_j^b \cong X[i] \oplus s_{j+1}[-k+1]^b$  and by [20, Lemma 3.7] it follows that  $\phi(s_j) = \phi(X[i]) = \phi(s_{j+1}[-k+1])$ ; if  $\beta = 0$ , then  $s_{j+1}[-k]^a \cong X[i] \oplus s_j[-1]^b$  and by [20, Lemma 3.7] it follows that  $\phi(s_{j+1}[-k]) = \phi(X[i]) = \phi(s_j[-1])$ . Thus we see that (143) implies  $P_\sigma^l = \{\pm \exp(i\pi\phi_\sigma(s_j)), \pm \exp(i\pi\phi_\sigma(s_{j+1}))\}$ , and (a) is proved.

(b) If  $\sigma \in \mathcal{Z}$ , then Lemma 7.8 shows that for any  $j \in \mathbb{Z}$  holds  $s_j, s_{j+1} \in \sigma^{ss}$ ,  $\phi_\sigma(s_j) < \phi_\sigma(s_{j+1}) < \phi_\sigma(s_j) + 1$ . Choosing one  $j \in \mathbb{Z}$ , denoting  $\sigma' = (Z', \mathcal{P}') = (-\log |Z(s_j)| + i\pi(1 - \phi_\sigma(s_j))) \star \sigma$  and using (32), (33), we get:

$$(145) \quad Z'(s_j) = -1, \quad |Z'(s_{j+1})| = \frac{|Z(s_{j+1})|}{|Z(s_j)|}, \quad \phi_{\sigma'}(s_{j+1}) = \phi_\sigma(s_{j+1}) + 1 - \phi_\sigma(s_j)$$

$$(146) \quad 1 = \phi_{\sigma'}(s_j) < \phi_{\sigma'}(s_{j+1}) < \phi_{\sigma'}(s_j) + 1 = 2 \Rightarrow 0 < \phi_{\sigma'}(s_{j+1}[-1]) < \phi_{\sigma'}(s_j) = 1.$$

Let  $\mathcal{A}$  be the extension closure of  $(s_j, s_{j+1}[-1])$ . Utilizing Lemma 7.5 and recalling that  $\text{hom}(s_j, s_{j+1}) = l \geq 2$  (see for example the arguments before (86)) we see that  $(s_j, s_{j+1}[-1])$  is an  $l$ -Kronecker pair [18, Definition 3.20], and by (146) it is a  $\sigma'$ -exceptional pair as well. From [20, Corollary 3.18] (and its proof) we see that the extension closure  $\mathcal{A}$  of  $(s_j, s_{j+1}[-1])$  coincides with  $\mathcal{P}'(0, 1]$ . Applying [18, Lemma 3.19] to  $(s_j, s_{j+1}[-1])$  we see that  $\mathcal{A}$  is the heart of a bounded t-structure of  $\mathcal{T}_l$  and due to the equality  $\mathcal{A} = \mathcal{P}'(0, 1]$  we have actually  $\sigma' \in \mathbb{H}^{\mathcal{A}}$  (see [18, Definition 2.28]). Now all the conditions of [18, Corollary 3.21] with the exceptional pair  $(s_j, s_{j+1}[-1])$  hold and we deduce that  $P_{\sigma'}^l = R_{v, \Delta_{l+}}$ , where  $v = (Z'(s_j), Z'(s_{j+1}[-1]))$ . On the other hand (45) shows that  $\exp(i\pi(1 - \phi_\sigma(s_j))) \cdot P_\sigma^l = R_{v, \Delta_l}$ .

To determine the set  $R_{v, \Delta_l}$  we use Lemma 7.24 and observe that now (see (145))

$v = \left(-1, \frac{|Z(s_{j+1})|}{|Z(s_j)|} \exp(i\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)))\right)$ ,  $0 < \pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)) < \pi$ , in particular the equality (141) yields (134) and the function (142) has the form (135).

(c) Let  $\sigma \in \mathcal{Z}$ . In (b)  $j$  was any integer, here we choose  $j = 0$ . Now formulas (145) and (146) give:

$$(147) \quad Z'(s_0) = -1 \quad Z'(s_1[-1]) = |Z'(s_1)| \exp(i\pi\phi_{\sigma'}(s_1[-1])), \quad 0 < \phi_{\sigma'}(s_1[-1]) < \phi_{\sigma'}(s_0) = 1.$$

Since  $s_0[1], s_1$  are the simple representations and since  $s_{\geq 1}, s_{\leq 0}[1] \in \text{Rep}_k(K(l))$  (Lemma 7.5), it follows that (see also (129)) for any  $i \geq 1$   $Z'(s_i) = n_i Z'(s_0[1]) + m_i Z'(s_1)$ , and for any  $i \leq 0$   $Z'(s_i[1]) = n_i Z'(s_0[1]) + m_i Z'(s_1)$ , and now using [19, Remark 3.16] (in particular  $f$  is as in (135))

we obtain :

$$(148) \quad \pm \frac{Z'(s_i)}{|Z'(s_i)|} = \begin{cases} \mp \frac{n_i Z'(s_0) + m_i Z'(s_1[-1])}{|n_i Z'(s_0) + m_i Z'(s_1[-1])|} = \mp \exp(\text{if}(n_i/m_i)) & i \geq 1 \\ \mp 1 & i = 0 \\ \pm \frac{n_i Z'(s_0) + m_i Z'(s_1[-1])}{|n_i Z'(s_0) + m_i Z'(s_1[-1])|} = \pm \exp(\text{if}(n_i/m_i)) & i \leq -1 \end{cases}.$$

In (b) we showed that  $P_{\sigma'}^l$  equals the set on the RHS of (134). Due to Lemma 7.22 we get that  $L(P_{\sigma'}^l) = \pm \exp(\text{if}([a_l^{-1}, a_l]))$ , and therefore (148) and (19) imply that  $P_{\sigma'}^l \setminus L(P_{\sigma'}^l) = \{\pm \exp(i\pi\phi_{\sigma'}(s_i))\}_{i \in \mathbb{Z}}$ . Recalling that  $\sigma' = \lambda \star \sigma$  for certain  $\lambda \in \mathbb{C}$  with the help of formulas (32) and (45) we deduce the desired  $P_{\sigma'}^l \setminus L(P_{\sigma'}^l) = \{\pm \exp(i\pi\phi_{\sigma}(s_i))\}_{i \in \mathbb{Z}}$ .

(d) In (b) we showed that  $P_{\sigma'}^l$  for  $\sigma' = (Z', \mathcal{P}') = (-\log|Z(s_j)| + i\pi(1 - \phi_{\sigma}(s_j))) \star \sigma$  equals the RHS of (134) and taking into account Lemma 7.22 we deduce that  $P_{\sigma'}^l \setminus L(P_{\sigma'}^l) = \pm 1 \cup \{\pm \exp(\text{if}(n_i/m_i)) : i \neq 0\}$ , which combined with (c) yields:

$$(149) \quad \pm 1 \cup \{\pm \exp(\text{if}(n_i/m_i)) : i \neq 0\} = \{\pm \exp(i\pi\phi_{\sigma'}(s_i))\}_{i \in \mathbb{Z}}.$$

Recalling that (133) holds for any  $j \in \mathbb{Z}$  and also (91), (146) we derive:

$$(150) \quad 0 = \phi_{\sigma'}(s_j) - 1 < \phi_{\sigma'}(s_{j+1}[-1]) < \phi_{\sigma'}(s_{j+2}[-1]) < \dots < \phi_{\sigma'}(s_{j-2}) < \phi_{\sigma'}(s_{j-1}) < \phi_{\sigma'}(s_j) = 1$$

We already know that (see (b) of the proposition and (e) in Lemma 7.22)

$$(151) \quad f(0) = f(n_1/m_1) = \pi(\phi_{\sigma}(s_{j+1}) - \phi_{\sigma}(s_j)) = \pi\phi_{\sigma'}(s_{j+1}[-1]) = \arccos(y).$$

Furthermore from (e) in Lemma 7.22 we deduce:

$$(152) \quad 0 < f\left(\frac{n_1}{m_1}\right) < f\left(\frac{n_2}{m_2}\right) < \dots < f(a_l^{-1}) \leq f(a_l) < \dots < f\left(\frac{n_{-2}}{m_{-2}}\right) < f\left(\frac{n_{-1}}{m_{-1}}\right) < \pi.$$

By induction the equalities (149), (150), (152) imply:

$$(153) \quad k \geq 1 \Rightarrow f\left(\frac{n_k}{m_k}\right) = \pi\phi_{\sigma'}(s_{j+k}[-1]) \quad f\left(\frac{n_{-k}}{m_{-k}}\right) = \pi\phi_{\sigma'}(s_{j-k}).$$

Now recalling that (see (33))

$$(154) \quad \forall i \in \mathbb{Z} \quad \phi_{\sigma'}(s_i) = \phi_{\sigma}(s_i) + 1 - \phi_{\sigma}(s_j)$$

we deduce (136), (137), (139) from (153), (151), and (150). The equality (138) in turn follows from (134), (153), (154).

(e) Follows easily from the already proven (d).

## 7.6. Computing the norms of Kornecker quivers, i. e. $\|D^b(K(l))\|_{\epsilon}$ .

If we define the function:

$$(155) \quad F : (0, +\infty) \times (-1, +1) \times (0, +\infty) \rightarrow (0, \pi) \quad F(x, y, t) = \arccos\left(\frac{xy - t}{\sqrt{t^2 + x^2 - 2txy}}\right)$$

then using Proposition 7.23 (a), Lemma 7.22 (e), and formulas (134), (135) one concludes that:



**Proposition 7.25.** *Let  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T}_l)$ .*

*If  $\sigma \notin \mathcal{Z}$ , then  $\text{vol}(\overline{P_\sigma^l}) = 0$ . If  $\sigma \in \mathcal{Z}$ , then for any  $j \in \mathbb{Z}$  holds:*

$$(156) \quad \frac{1}{2} \text{vol}(\overline{P_\sigma^l}) = F(x_j(\sigma), y_j(\sigma), a_l) - F(x_j(\sigma), y_j(\sigma), a_l^{-1}),$$

where  $x_j(\sigma) = \frac{|Z(s_{j+1})|}{|Z(s_j)|} \quad y_j(\sigma) = \cos(\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j))).$

One computes

$$(157) \quad \frac{\partial}{\partial x} F(x, y, t) = \frac{-t\sqrt{1-y^2}}{t^2 + x^2 - 2txy} \quad \frac{\partial}{\partial t} F(x, y, t) = \frac{x\sqrt{1-y^2}}{t^2 + x^2 - 2txy}$$

and therefore:

$$(158) \quad \frac{\partial}{\partial x} (F(x, y, a_l) - F(x, y, a_l^{-1})) = \frac{a_l(a_l^2 - 1)\sqrt{1-y^2}}{(1 + (a_l x)^2 - 2a_l x y)(a_l^2 + x^2 - 2a_l x y)}(1 - x^2),$$

which implies that for any  $x > 0$ ,  $y \in (-1, +1)$  we have:

$$(159) \quad F(x, y, a_l) - F(x, y, a_l^{-1}) \leq F(1, y, a_l) - F(1, y, a_l^{-1})$$

On the other hand one computes that for any  $y \in (-1, +1)$ ,  $t \in (0, +\infty)$  holds:

$$(160) \quad F(1, y, a_l) - F(1, y, a_l^{-1}) = \arccos\left(\frac{y - a_l}{\sqrt{a_l^2 + 1 - 2a_l y}}\right) - \arccos\left(\frac{a_l y - 1}{\sqrt{a_l^2 + 1 - 2a_l y}}\right)$$

$$(161) \quad \frac{\partial}{\partial y} (F(1, y, a_l) - F(1, y, a_l^{-1})) = \begin{cases} \frac{a_l^2 - 1}{\sqrt{1-y^2}(a_l^2 + 1 - 2a_l y)} > 0 & l \geq 3 \\ 0 & l = 2 \end{cases}$$

$$(162) \quad \frac{\partial}{\partial t} (F(1, y, t) - F(1, y, t^{-1})) = \frac{2\sqrt{1-y^2}}{t^2 + 1 - 2ty} > 0.$$

Therefore the numbers (163) depending on  $\varepsilon \in (0, 1)$  and  $l \geq 2$  satisfy (164), (165), (166):

$$(163) \quad K_\varepsilon(l) = \arccos\left(\frac{\cos(\pi\varepsilon) - a_l}{\sqrt{a_l^2 + 1 - 2a_l \cos(\pi\varepsilon)}}\right) - \arccos\left(\frac{a_l \cos(\pi\varepsilon) - 1}{\sqrt{a_l^2 + 1 - 2a_l \cos(\pi\varepsilon)}}\right)$$

$$(164) \quad 0 < \varepsilon < 1 \quad \Rightarrow \quad K_\varepsilon(2) = 0$$

$$(165) \quad l \in \mathbb{N}_{\geq 3} \quad 0 < u < v < +1 \quad \Rightarrow \quad K_u(l) > K_v(l)$$

$$(166) \quad 0 < \varepsilon < 1 \quad 2 \leq l_1 < l_2 \in \mathbb{N}_{\geq 2} \quad \Rightarrow \quad K_\varepsilon(l_1) < K_\varepsilon(l_2)$$

$$(167) \quad \lim_{l \rightarrow +\infty} K_\varepsilon(l) = \pi(1 - \varepsilon).$$

The inequality (159) and the derivative (161) imply that for  $\varepsilon \in (0, +1)$  and  $l \geq 2$  holds:

$$(168) \quad \sup_{(x,y) \in (0, +\infty) \times (-1, \cos(\pi\varepsilon))} \{F(x, y, a_l) - F(x, y, a_l^{-1})\} = K_\varepsilon(l),$$

Note that  $\sup_{(x,y) \in (0, +\infty) \times (-1, 1)} \{F(x, y, a_l) - F(x, y, a_l^{-1})\}$  is always equal to  $\pi$  independently on  $l \geq 3$  as opposed to  $K_\varepsilon(l)$ , which is strictly increasing on  $l$ .

Finally we note that for  $\varepsilon = 1/2$  the expression (163) takes a simple form (recall that  $l = \frac{a_l^2+1}{a_l}$ ):<sup>12</sup>

$$(169) \quad K_{\frac{1}{2}}(l) = \arccos\left(\frac{2}{l}\right).$$

Now we can compute  $\|D^b(K(l))\|_\varepsilon$ .

**Proposition 7.26.** *Let  $\varepsilon \in (0, 1)$ ,  $l \geq 2$ , and let  $K_\varepsilon(l)$  be as in (163). Then  $\|D^b(K(l))\|_\varepsilon = K_\varepsilon(l)$ .*

*Proof.* From Proposition 7.23 (a), (b), (e) we see that  $P_\sigma$  is not dense in  $\mathbb{S}^1$  for all  $\sigma$ , and (48) reduces to the following formula:

$$\begin{aligned} \|D^b(K(l))\|_\varepsilon &= \sup \left\{ \frac{\text{vol}(\overline{P_\sigma^l})}{2} : \sigma \in \mathcal{Z} \text{ and there exists } j \in \mathbb{Z} \text{ such that } \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon \right\} \\ &= \sup \left\{ \frac{\text{vol}(\overline{P_\sigma^l})}{2} : \sigma \in \bigcup_{j \in \mathbb{Z}} \{x \in \mathcal{Z} : \phi_x(s_{j+1}) - \phi_x(s_j) > \varepsilon\} \right\} \\ &= \sup \left\{ \sup \left\{ \frac{\text{vol}(\overline{P_\sigma^l})}{2} : \sigma \in \mathcal{Z} \text{ and } \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon \right\} : j \in \mathbb{Z} \right\}. \end{aligned}$$

By using (156) and (168) we will show that for all  $j \in \mathbb{Z}$  holds:

$$(170) \quad \sup \left\{ \frac{\text{vol}(\overline{P_\sigma^l})}{2} : \sigma \in \mathcal{Z} \text{ and } \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon \right\} = K_\varepsilon(l)$$

and then the proposition follows. Recalling Lemma 7.8 we see that  $\sigma \in \mathcal{Z}$  and  $\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon$  iff  $s_j, s_{j+1} \in \sigma^{ss}$  and  $\varepsilon < \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) < 1$ , furthermore restricting the map (93) to the set of stability conditions  $\sigma$  with  $s_j, s_{j+1} \in \sigma^{ss}$  and  $\varepsilon < \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) < 1$  we see that the set of pairs  $(x_j(\sigma), y_j(\sigma))$  from (156) for these  $\sigma$  is:

$$(171) \quad \{(x_j(\sigma), y_j(\sigma)) : \sigma \in \mathcal{Z} \text{ and } \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon\} = (0, +\infty) \times (-1, \cos(\pi\varepsilon)).$$

Combining the latter equality with (156) and (168) leads to (170).  $\square$

**Corollary 7.27.** *Let  $\varepsilon \in (0, 1)$ . Let  $2 \leq l_1$  and  $2 \leq l_2$ . Then:*

$$(172) \quad \|D^b(K(l_1))\|_\varepsilon < \|D^b(K(l_2))\|_\varepsilon \iff l_1 < l_2$$

*Proof.* Follows from the previous proposition and (166).  $\square$

**Corollary 7.28.** *Let  $\varepsilon \in (0, 1)$  and let  $l \in \mathbb{N}_{\geq 1}$ .*

*Then  $\|D^b(K(l))\|_\varepsilon = 0$  iff  $\text{Stab}(D^b(K(l)))$  is affine (biholomorphic to  $\mathbb{C}^2$ ).*

*Proof.* Propositions 4.19 and 7.26 imply that  $\|D^b(K(l))\|_\varepsilon = 0$  iff  $l \leq 2$ , and table (9) shows that  $\text{Stab}(D^b(K(l)))$  is biholomorphic to  $\mathbb{C}^2$  iff  $l \leq 2$ .  $\square$

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<sup>12</sup>one shows this using the equality  $\arccos(x) - \arccos(\sqrt{1-x^2}) = \arccos(2x\sqrt{1-x^2})$ , which holds for  $0 \leq x \leq \frac{1}{\sqrt{2}}$

8. THE INEQUALITY  $\|\langle E_1, E_2 \rangle\|_\varepsilon \geq K_\varepsilon(\text{hom}^{\min}(E_1, E_2))$ 

In this section we derive a formula, which will help us to compute other norms. To that end it is useful to extend the definition of  $K_\varepsilon(l)$  in (163) by postulating  $K_\varepsilon(0) = K_\varepsilon(1) = 0$ . Recall that the notation  $\text{hom}^{\min}(E_1, E_2)$  is explained in (15).

**Proposition 8.1.** *Let  $\mathcal{T}$  be a proper category, and let  $(E_1, E_2)$  be any exceptional pair in it. Then*

$$(173) \quad \|\langle E_1, E_2 \rangle\|_\varepsilon \geq K_\varepsilon(\text{hom}^{\min}(E_1, E_2)) \quad \text{for } \varepsilon \in (0, 1).$$

*Proof.* We can assume that  $\text{hom}^{\leq 0}(E_1, E_2) = 0$  and  $l = \text{hom}^1(E_1, E_2) \neq 0$ , and under these assumption we have to show that

$$(174) \quad \|\langle E_1, E_2 \rangle\|_\varepsilon \geq K_\varepsilon(l).$$

Let  $\mathcal{D}$  be the triangulated subcategory  $\langle E_1, E_2 \rangle$ . The assumptions on  $(E_1, E_2)$  are the same as in the definition of an  $l$ -Kronecker pair, [18, Definition 3.20], and we can apply [18, Lemma 3.19, Corollary 3.21] to it. In particular the extension closure  $\mathcal{A}$  of  $(E_1, E_2)$  is a heart of a bounded t-structure in  $\mathcal{D}$  with simple objects  $E_1, E_2$ , and any stability condition  $\sigma = (Z, \mathcal{P}) \in \mathbb{H}^{\mathcal{A}} \subset \text{Stab}(\mathcal{D})$  with  $\arg(Z(E_1)) > \arg(Z(E_2))$  satisfies  $P_\sigma^{\mathcal{D}} = R_{v, \Delta_{l+}}$ , where  $v = (Z(E_1), Z(E_2))$ . The arguments in the beginning of the proof of Lemma 4.7 show that for each  $v \in \mathbb{H}^2$  there exists unique  $\sigma = (Z, \mathcal{P}) \in \mathbb{H}^{\mathcal{A}}$  with  $v = (Z(E_1), Z(E_2))$  and that  $\sigma$  is full. For any  $0 < \mu$  such that  $\mu + \varepsilon < 1$  choose the vector  $(-1, \exp(i\pi(\varepsilon + \mu))) = v_\mu$  and denote by  $\sigma_\mu$  the stability condition  $\sigma_\mu = (\mathcal{P}_\mu, Z_\mu) \in \mathbb{H}^{\mathcal{A}}$  with  $(Z_\mu(E_1), Z_\mu(E_2)) = v_\mu$ . The given arguments ensure that  $\sigma_\mu$  is full and  $P_{\sigma_\mu}^{\mathcal{D}} = R_{v_\mu, \Delta_{l+}}$ . Using the formula for  $R_{v_\mu, \Delta_{l+}}$  in Lemma 7.24 for the given  $v_\mu$  one derives:

$$(175) \quad \frac{\text{vol}(\overline{P_{\sigma_\mu}^{\mathcal{D}}})}{2} = \frac{\text{vol}(\overline{R_{v_\mu, \Delta_{l+}}})}{2} = K_{\varepsilon+\mu}(l),$$

where  $K_{\varepsilon+\mu}(l)$  is in (163). Note that the arc  $\exp(i\pi[\mu/2, \varepsilon + \mu/2])$  is in the complement of  $P_{\sigma_\mu}^{\mathcal{D}}$  and therefore  $\sigma_\mu \in \text{Stab}_\varepsilon(\mathcal{D})$ . Now from the very Definition 4.11 we see that  $\|\mathcal{D}\|_\varepsilon \geq K_{\varepsilon+\mu}(l)$  for any small enough positive  $\mu$ , letting  $\mu \rightarrow 0$  we derive the desired  $\|\mathcal{D}\|_\varepsilon \geq K_\varepsilon(l)$ .  $\square$

**Corollary 8.2.** *Let  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  be an exceptional collection in a proper triangulated category  $\mathcal{T}$ . Then for any  $0 \leq i < j \leq n$  we have  $\|\langle \mathcal{E} \rangle\|_\varepsilon \geq K_\varepsilon(\text{hom}^{\min}(E_i, E_j))$ .*

*Proof.* Take  $0 \leq i < j \leq n$ . By mutating the sequence  $\mathcal{E}$  (see Remark 7.3) one can get a sequence  $\mathcal{E}'$  of the form  $\mathcal{E}' = (E_i, E_j, C_2, \dots, C_n)$  such that  $\langle \mathcal{E} \rangle = \langle \mathcal{E}' \rangle$ . Corollary 6.2 implies  $\|\langle \mathcal{E} \rangle\|_\varepsilon = \|\langle \mathcal{E}' \rangle\|_\varepsilon \geq \|\langle E_i, E_j \rangle\|_\varepsilon$ , and due to Proposition 8.1 we get  $\|\langle E_i, E_j \rangle\|_\varepsilon \geq K_\varepsilon(\text{hom}^{\min}(E_i, E_j))$ .  $\square$

**Corollary 8.3.** *Let  $\mathcal{T}$  be a proper triangulated category such that for each  $l \in \mathbb{N}$  there exists a full exceptional collection  $(E_0, E_1, \dots, E_n)$  and integers  $0 \leq i < j \leq n$  for which  $\text{hom}^{\min}(E_i, E_j) \geq l$ . Then  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$  for any  $\varepsilon \in (0, 1)$ .*

*Proof.* The given property of  $\mathcal{T}$  combined with Corollary 8.2 amounts to  $\|\mathcal{T}\|_\varepsilon \geq K_\varepsilon(l)$  for each  $l \geq \mathbb{N}$  (Recall also (166)). Now from (167) and Remark 4.14 we obtain  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ .  $\square$

**Corollary 8.4.** *Let  $\mathcal{T}$  be a proper category, and let  $0 < \varepsilon < 1$ .*

(a) *If  $\|\mathcal{T}\|_\varepsilon = 0$ , then for any full exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  and for any  $0 \leq i < j \leq n$  we have  $\text{hom}^{\min}(E_i, E_j) \leq 2$ .*

(b) If  $\|\mathcal{T}\|_\varepsilon \leq K_\varepsilon(l)$ ,  $l \geq 2$ , then for any full exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  and for any  $0 \leq i < j \leq n$  we have  $\text{hom}^{\min}(E_i, E_j) \leq l$ .

(c) If  $\|\mathcal{T}\|_\varepsilon < \pi(1 - \varepsilon)$ , then there exists  $l \in \mathbb{N}$  such that for any full exceptional collection  $\mathcal{E} = (E_0, E_1, \dots, E_n)$  and for any  $0 \leq i < j \leq n$  we have  $\text{hom}^{\min}(E_i, E_j) \leq l$ .

We will apply Corollary 8.3 to various examples.

**Definition 8.5.** For any quiver  $Q$  and any subset  $A \subset V(Q)$  we denote by  $Q_A$  the quiver whose vertices are  $A$  and whose arrows are those arrows of  $Q$  whose initial and final vertex is in  $A$ . A vertex  $v \in V(Q)$  is called adjacent to  $A$  if there exists an arrow in  $Q$  starting at  $v$  and ending at a vertex of  $A$  or an arrow starting at a vertex of  $A$  and ending at a  $v$ .

**Corollary 8.6.** Let  $Q$  be an acyclic quiver. If there exists a subset  $A \subset V(Q)$  such that the quiver  $Q_A$  is affine and a vertex  $v \in V(Q)$  such that  $v$  is a source or a sink in  $Q_{A \cup \{v\}}$ , then  $\|D^b(Q)\|_\varepsilon = \pi(1 - \varepsilon)$  for any  $\varepsilon \in (0, 1)$ .

*Proof.* Let  $l \geq 3$ . By [18, Corollary 3.36] for any  $l \geq 3$  there exists a an exceptional pair  $(E_0, E_1)$  in  $D^b(Q)$  such that  $\text{hom}^{\min}(E_0, E_1) \geq l$ . In [17] is shown that  $(E_0, E_1)$  can be extended to a full exceptional collection. Therefore we can apply Corollary 8.3 to  $D^b(Q)$ .  $\square$

Now we present one method (Lemma 8.8) to obtain  $l$ -Kronecker pairs with arbitrary big  $l$  as part of full exceptional collections, i.e. method to obtain the conditions of Corollary 8.3. This method relies on full exceptional collections in which a triple remains strong after certain mutations (see (c) in the statement of Lemma 8.8). In [9] a strong exceptional collection  $\mathcal{E}$  which remains strong under all mutations is called *non-degenerate*. Furthermore in [9] are defined so called *geometric* exceptional collections and [9, Corollary 2.4] says that geometricity implies non-degeneracy. Furthermore, [9, Proposition 3.3] claims that a full exceptional collection of length  $m$  of coherent sheaves on a smooth projective variety  $X$  of dimension  $n$  is geometric if and only if  $m = n + 1$ . In particular it follows:

**Remark 8.7.** The full exceptional collection  $\mathcal{E} = \{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\}$  in  $D^b(\mathbb{P}^n)$  introduced by Beilinson [5] is geometric and therefore non-degenerate, whereas the well known (see [45], [27]) strong full exceptional collection of line bundles  $(\mathcal{O}(0, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 0), \mathcal{O}(1, 1))$  in  $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$  is not geometric.

That's why the method of Lemma 8.8 is readily applied to  $D^b(\mathbb{P}^n)$  in Corollary 8.9, whereas applying it to  $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$  in Corollary 8.10 requires some additional arguments to ensure (c).

**Lemma 8.8.** Let  $\mathcal{T}$  be a proper triangulated category and  $\varepsilon \in (0, 1)$ . Let  $\mathcal{E} = (F_0, F_1, F_2, E_3, \dots, E_n)$  be a full exceptional collection with  $n \geq 3$ . Let  $\{F_i\}_{i \in \mathbb{N}}$  be a sequence starting with  $F_0, F_1, F_2$  and  $F_{i+1} = R_{F_i}(F_{i-1})$  for  $i \geq 2$ . If the following three properties hold:

(a)  $\text{hom}(F_0, F_1) < \text{hom}(F_0, F_2)$ ; (b)  $l = \text{hom}(F_1, F_2) \geq 2$ ; (c)  $(F_0, F_i, F_{i+1})$  is strong for all  $i \geq 1$ , then  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ .

*Proof.* Now (83) becomes

$$(176) \quad F_{i-1} \xrightarrow{\text{coev}_{F_{i-1}, F_i}^*} \text{Hom}^*(F_{i-1}, F_i)^\vee \otimes F_i \longrightarrow R_{F_i}(F_{i-1}) = F_{i+1} \quad i \geq 2.$$

Since the property of being full is preserved under mutations, it follows that  $(F_0, F_{i-1}, F_i, E_3, \dots, E_n)$  is full for each  $i \geq 2$ . We will show that (177) holds, and then our  $\mathcal{T}$  satisfies the conditions of

Corollary 8.3, hence  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$ .

$$(177) \quad i \in \mathbb{N}_{\geq 2} \Rightarrow \operatorname{hom}(F_0, F_{i-1}) < \operatorname{hom}(F_0, F_i)$$

To show (177) we first note that due to (c) we have  $\operatorname{hom}^k(F_{i-1}, F_i) = 0$  for each  $k \neq 0$  and each  $i \geq 2$  and it follows that (see e.g. [9, Example 2.7])  $l = \operatorname{hom}(F_1, F_2) = \operatorname{hom}(F_{i-1}, F_i) = \operatorname{hom}(F_i, F_{i+1})$  for each  $i \geq 2$  and then (176) has the form:

$$(178) \quad F_{i-1} \xrightarrow{\operatorname{coev}_{F_{i-1}, F_i}^*} F_i^{\oplus l} \longrightarrow F_{i+1} \quad i \geq 2.$$

In (a) we are given  $\operatorname{hom}(F_0, F_{i-1}) < \operatorname{hom}(F_0, F_i)$  for  $i = 2$  and we will show (177) by induction. Indeed, since  $(F_0, F_{i-1}, F_i)$  is a strong exceptional collection for each  $i \geq 2$ , applying  $\operatorname{Hom}(F_0, -)$  to (178) yields short exact sequences between finite dimensional vector spaces:

$$(179) \quad 0 \longrightarrow \operatorname{Hom}(F_0, F_{i-1}) \longrightarrow \operatorname{Hom}(F_0, F_i)^{\oplus l} \longrightarrow \operatorname{Hom}(F_0, F_{i+1}) \longrightarrow 0, \quad i \geq 2.$$

The obtained exact sequences and  $l \geq 2$  imply:

$$(180) \quad \begin{aligned} \operatorname{hom}(F_0, F_{i+1}) &= l \operatorname{hom}(F_0, F_i) - \operatorname{hom}(F_0, F_{i-1}) \geq 2 \operatorname{hom}(F_0, F_i) - \operatorname{hom}(F_0, F_{i-1}) \\ &= \operatorname{hom}(F_0, F_i) + (\operatorname{hom}(F_0, F_i) - \operatorname{hom}(F_0, F_{i-1})), \end{aligned}$$

hence for  $i \geq 2$  the inequality  $\operatorname{hom}(F_0, F_i) > \operatorname{hom}(F_0, F_{i-1})$  implies  $\operatorname{hom}(F_0, F_{i+1}) > \operatorname{hom}(F_0, F_i)$ . The lemma is proved.  $\square$

**Corollary 8.9.** *For  $n \geq 2$  and  $\varepsilon \in (0, 1)$  we have  $\|D^b(\mathbb{P}^n)\|_\varepsilon = \pi(1 - \varepsilon)$ .*

*Proof.* In Remark 8.7 is given a full strong exceptional collection  $\mathcal{E}$  on  $D^b(\mathbb{P}^n)$  which remains strong under all mutations. Using [9, Example 2.9] one computes  $\operatorname{hom}(\mathcal{O}, \mathcal{O}(1)) = \operatorname{hom}(\mathcal{O}(1), \mathcal{O}(2)) = n + 1 < \operatorname{hom}(\mathcal{O}, \mathcal{O}(2)) = \frac{(n+1)(n+2)}{2}$ . Therefore we can apply Lemma 8.8 and the corollary follows.  $\square$

**Corollary 8.10.** *For each  $\varepsilon \in (0, 1)$  we have  $\|D^b(\mathbb{P}^1 \times \mathbb{P}^1)\|_\varepsilon = \pi(1 - \varepsilon)$ .*

*Proof.* Let us denote here  $\mathcal{T} = D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ . Exceptional collections in  $\mathcal{T}$  have been studied in [45] and [27]. In particular the full strong exceptional collection  $(\mathcal{O}(0, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 0), \mathcal{O}(1, 1))$  mentioned in Remark 8.7 satisfies  $\operatorname{hom}(\mathcal{O}(0, 0), \mathcal{O}(0, 1)) = \operatorname{hom}(\mathcal{O}(0, 1), \mathcal{O}(1, 1)) = 2$  and  $\operatorname{hom}(\mathcal{O}(0, 0), \mathcal{O}(1, 1)) = 4$  (see [41, p. 3] or [6, Example 6.5]). After one mutation we get a full exceptional collection  $(F_0, F_1, F_2, E_3)$  in which  $(F_0, F_1, F_2)$  is strong,  $\operatorname{hom}(F_0, F_1) < \operatorname{hom}(F_0, F_2)$ , and  $\operatorname{hom}(F_1, F_2) = 2$ . Let  $\{F_i\}_{i \in \mathbb{N}}$  be a sequence starting with  $F_0, F_1, F_2$  and  $F_{i+1} = R_{F_i}(F_{i-1})$  for  $i \geq 2$ . To apply Lemma 8.8 and deduce that  $\|\mathcal{T}\|_\varepsilon = \pi(1 - \varepsilon)$  we need to show that  $(F_0, F_i, F_{i+1})$  is strong for all  $i \geq 1$ .

From [27, Proposition 5.3.1, Theorem 3.3.1.] it follows that:

$$(181) \quad \text{For each exceptional pair } (E, F) \text{ in } \mathcal{T} \text{ there is at most one } i \in \mathbb{Z} \text{ with } \operatorname{hom}^i(E, F) \neq 0.$$

From the way we defined  $\{F_i\}_{i \in \mathbb{N}}$  it follows (see e.g. [9, Example 2.7])  $2 = \operatorname{hom}(F_1, F_2) = \operatorname{hom}(F_{i-1}, F_i) = \operatorname{hom}(F_i, F_{i+1})$  for all  $i \geq 2$ , hence taking into account (181), to show that  $(F_0, F_i, F_{i+1})$  is strong for all  $i \geq 1$  suffices to show that  $\operatorname{hom}(F_0, F_i) \neq 0$  for each  $i \geq 1$ . Now (83) becomes distinguished triangle

$$(182) \quad F_{i-1} \xrightarrow{\operatorname{coev}_{F_{i-1}, F_i}^*} F_i^{\oplus 2} \longrightarrow F_{i+1} \longrightarrow F_{i-1}[1] \quad i \geq 2.$$

We have  $0 < \text{hom}(F_0, F_1) < \text{hom}(F_0, F_2)$ . Assume that for some  $i \geq 2$  holds

$$(183) \quad 0 < \text{hom}(F_0, F_1) < \cdots < \text{hom}(F_0, F_{i-1}) < \text{hom}(F_0, F_i)$$

we will show that this implies  $\text{hom}(F_0, F_i) < \text{hom}(F_0, F_{i+1})$  and by induction the corollary follows. Applying  $\text{Hom}(F_0, -)$  to (182) and since  $\text{hom}^k(F_0, F_{i-1}) = \text{hom}^k(F_0, F_i) = 0$  for  $k \neq 0$  one easily deduces that  $\text{hom}^k(F_0, F_{i+1}) = 0$  for  $k \notin \{-1, 0\}$ . If  $\text{hom}^{-1}(F_0, F_{i+1}) \neq 0$ , then by (181) it follows that  $\text{hom}(F_0, F_{i+1}) = 0$  and applying  $\text{Hom}(F_0, -)$  to (182) yields an exact sequence of vector spaces:

$$(184) \quad 0 \longrightarrow \text{Hom}^{-1}(F_0, F_{i+1}) \longrightarrow \text{Hom}(F_0, F_{i-1}) \longrightarrow \text{Hom}(F_0, F_i)^{\oplus 2} \longrightarrow \text{Hom}(F_0, F_{i+1}) = 0,$$

which contradicts (183). Therefore  $\text{hom}^{-1}(F_0, F_{i+1}) = 0$  and  $\text{hom}^k(F_0, F_{i+1}) = 0$  for  $k \neq 0$ . Now we apply  $\text{Hom}(F_0, -)$  to (182) again and get a short exact sequence as in (178) which by the same computation as in (180) implies  $\text{hom}(F_0, F_{i+1}) > \text{hom}(F_0, F_i)$ , thus we proved the corollary.  $\square$

**Corollary 8.11.** *Let  $X$  be a smooth algebraic variety obtained from  $\mathbb{P}^n$ ,  $n \geq 2$ , or from  $\mathbb{P}^1 \times \mathbb{P}^1$  by blowing up finite number (possibly zero) of points. Then  $\|D^b(X)\|_\varepsilon = \pi(1 - \varepsilon)$ .*

*Proof.* This follows from Corollaries 6.3, 8.9, 8.10.  $\square$

## 9. THE INEQUALITY $\|\mathcal{T}_{l_1} \oplus \cdots \oplus \mathcal{T}_{l_n}\|_\varepsilon < \pi(1 - \varepsilon)$

The goal of this section is to prove the following:

**Proposition 9.1.** *Let  $n \geq 1$ , let  $l_i \geq 1$ ,  $i = 1, 2, \dots, n$  be a sequence of integers, and let  $0 < \varepsilon < 1$ . Then for any orthogonal decomposition of the form  $\mathcal{T} = \mathcal{T}_{l_1} \oplus \mathcal{T}_{l_2} \oplus \cdots \oplus \mathcal{T}_{l_n}$ , where  $\mathcal{T}_{l_i} \cong D^b(K(l_i))$ , holds  $\|\mathcal{T}\|_\varepsilon < \pi(1 - \varepsilon)$ . Furthermore  $\|\mathcal{T}\|_\varepsilon > 0$  iff  $l_i \geq 3$  for some  $1 \leq i \leq n$ .*

Before going to the proof of this proposition we prove some facts for the case  $l \geq 3$  and denote  $\mathcal{T}_l = D^b(K(l))$ . We will use notations and results from Section 7. The first step is:

**Lemma 9.2.** *For  $\sigma \notin \mathcal{Z}$  the set  $P_\sigma^l = \overline{P_\sigma^l}$  is finite. Otherwise, for  $\sigma \in \mathcal{Z}$ , we use the description of the set  $\overline{P_\sigma^l}$  as in Proposition 7.23 (136), (137), (138).*

*For any  $0 < \varepsilon < 1$  there exists  $M_{l,\varepsilon} > 0$  such that for any  $\sigma \in \mathcal{Z}$  and for any  $j \in \mathbb{Z}$  :*

$$(185) \quad \phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon \quad \Rightarrow \quad \frac{v_\sigma - u_\sigma}{u_\sigma - \pi\phi_\sigma(s_{j+1}[-1])} \leq M_{\varepsilon,l} \quad \frac{v_\sigma - u_\sigma}{\pi\phi_\sigma(s_j) - v_\sigma} \leq M_{\varepsilon,l}.$$

*Proof.* The part of the lemma, which is not contained in Proposition 7.23 are the inequalities (185). So, let us chose  $\sigma \in \mathcal{Z}$ ,  $j \in \mathbb{Z}$  and  $0 < \varepsilon < 1$  and assume that  $\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) > \varepsilon$ . In terms of the function (155) we can rewrite (139) as follows:

$$(186) \quad \frac{v_\sigma - u_\sigma}{u_\sigma - \pi\phi_\sigma(s_{j+1}[-1])} = \frac{F(x, y, a_l) - F(x, y, a_l^{-1})}{F(x, y, a_l^{-1}) - \arccos(y)}; \quad \frac{v_\sigma - u_\sigma}{\pi\phi_\sigma(s_j) - v_\sigma} = \frac{F(x, y, a_l) - F(x, y, a_l^{-1})}{\pi - F(x, y, a_l)}$$

where (recall that  $\sigma \in \mathcal{Z}$  implies  $\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j) < 1$ ):

$$(187) \quad 0 < x = \frac{|Z(s_{j+1})|}{|Z(s_j)|} \quad -1 < y = \cos(\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j))) < \cos(\pi\varepsilon) :$$

For any  $a \in (0, +\infty)$ ,  $b \in (-1, +1)$  the differentiable functions  $(0, +\infty) \ni t \mapsto F(a, b, t)$  and  $(0, +\infty) \ni t \mapsto F(t, b, a)$  can be extended uniquely to continuous functions in  $[0, +\infty)$  having values  $\arccos(b)$  and  $\pi$  at 0, respectively, and therefore we can apply the mean value theorem to these functions. More precisely, if  $h : [0, +\infty) \rightarrow \mathbb{R}$  is a function obtained in such a way, then for

any  $0 \leq \alpha < \beta < +\infty$  there exists  $\alpha < t < \beta$ , such that  $h(\beta) - h(\alpha) = (\beta - \alpha)h'(t)$ . In particular, for any  $x, y$  as in (187) we can represent all the differences in (186) as follows (recall (157)):

$$(188) \quad \frac{1}{\pi - F(x, y, a_l)} = \frac{-1}{F(x, y, a_l) - \pi} = \frac{1}{x \frac{a_l \sqrt{1-y^2}}{a_l^2 + x'^2 - 2a_l x' y}} = \frac{a_l^2 + x'^2 - 2a_l x' y}{x a_l \sqrt{1-y^2}} \text{ for some } 0 < x' < x$$

$$(189) \quad \frac{1}{F(x, y, a_l^{-1}) - \arccos(y)} = \frac{1}{a_l^{-1} \frac{x \sqrt{1-y^2}}{t'^2 + x^2 - 2t' x y}} = \frac{a_l(t^2 + x^2 - 2txy)}{x \sqrt{1-y^2}} \text{ for some } 0 < t < a_l^{-1}$$

$$(190) \quad F(x, y, a_l) - F(x, y, a_l^{-1}) = \frac{(a_l - a_l^{-1})x \sqrt{1-y^2}}{t'^2 + x^2 - 2t' x y} \leq \frac{(a_l - a_l^{-1})x \sqrt{1-y^2}}{t'^2 + x^2 - 2t' x \cos(\pi\varepsilon)} \text{ for some } a_l^{-1} < t' < a_l.$$

And now looking back at (186) we deduce:

$$(191) \quad \frac{v_\sigma - u_\sigma}{u_\sigma - \pi\phi_\sigma(s_{j+1}[-1])} \leq \frac{(a_l^2 - 1)(t^2 + x^2 - 2txy)}{t'^2 + x^2 - 2t' x \cos(\pi\varepsilon)}$$

$$(192) \quad \frac{v_\sigma - u_\sigma}{\pi\phi_\sigma(s_j) - v_\sigma} \leq \frac{(1 - a_l^{-2})(a_l^2 + x'^2 - 2a_l x' y)}{t'^2 + x^2 - 2t' x \cos(\pi\varepsilon)}.$$

Now since  $t'^2 + x^2 - 2t' x \cos(\pi\varepsilon)$  gets minimal values for  $t' = x \cos(\pi\varepsilon)$  (with respect to the variable  $t'$ ) and for  $x = t' \cos(\pi\varepsilon)$  (with respect to the variable  $x$ ) we have  $t'^2 + x^2 - 2t' x \cos(\pi\varepsilon) \geq x^2(1 - \cos^2(\pi\varepsilon)) = x^2 \sin^2(\pi\varepsilon)$  and  $t'^2 + x^2 - 2t' x \cos(\pi\varepsilon) \geq t'^2 \sin^2(\pi\varepsilon) \geq a_l^{-2} \sin^2(\pi\varepsilon)$ , therefore:

$$(193) \quad t'^2 + x^2 - 2t' x \cos(\pi\varepsilon) \geq \max\{a_l^{-2}, x^2\} \sin^2(\pi\varepsilon)$$

and (191), (192) can be continued (recall that  $0 < t < a_l^{-1}$  in (189) and  $0 < x' < x$  in (188)):

$$\begin{aligned} \frac{v_\sigma - u_\sigma}{u_\sigma - \pi\phi_\sigma(s_{j+1}[-1])} &\leq \frac{(a_l^2 - 1)(t^2 + x^2 - 2txy)}{\max\{a_l^{-2}, x^2\} \sin^2(\pi\varepsilon)} \leq \frac{(a_l^2 - 1)}{\sin^2(\pi\varepsilon)} \sup \left\{ \frac{(t^2 + x^2 - 2txy)}{\max\{a_l^{-2}, x^2\}} : \begin{array}{l} t \in (0, a_l^{-1}) \\ x \in (0, +\infty) \\ y \in (-1, \cos(\pi\varepsilon)) \end{array} \right\} \\ \frac{v_\sigma - u_\sigma}{\pi\phi_\sigma(s_j) - v_\sigma} &\leq \frac{(1 - a_l^{-2})(a_l^2 + x'^2 - 2a_l x' y)}{\max\{a_l^{-2}, x^2\} \sin^2(\pi\varepsilon)} \leq \frac{(1 - a_l^{-2})}{\sin^2(\pi\varepsilon)} \sup \left\{ \frac{(a_l^2 + x'^2 - 2a_l x' y)}{\max\{a_l^{-2}, x^2\}} : \begin{array}{l} x \in (0, +\infty) \\ x' \in (0, x) \\ y \in (-1, \cos(\pi\varepsilon)) \end{array} \right\} \end{aligned}$$

hence (185) follows.  $\square$

**Corollary 9.3.** For any  $\sigma \in \mathcal{Z}$  there is closed  $\frac{\text{vol}(\overline{P_\sigma^l})}{2}$ -arc  $p_\sigma^l \subset \overline{P_\sigma^l}$  s.t.  $\overline{P_\sigma^l} \setminus (p_\sigma^l \cup -p_\sigma^l)$  is countable.

Let  $0 < \varepsilon < 1$ . For any closed  $\varepsilon$ -arc  $\gamma$  satisfying  $P_\sigma^l \cap \gamma = \emptyset$  hold  $(p_\sigma^l \cup -p_\sigma^l) \cap (\gamma \cup -\gamma) = \emptyset$  and any (of the four) connected component  $c$  of  $\mathbb{S}^1 \setminus \{p_\sigma^l \cup -p_\sigma^l \cup \gamma \cup -\gamma\}$  restricts  $\text{vol}(\overline{P_\sigma^l})$  as follows:

$$(194) \quad c \subset \mathbb{S}^1 \setminus \{p_\sigma^l \cup -p_\sigma^l \cup \gamma \cup -\gamma\} \quad \pi_0(c) = \{0\} \quad \Rightarrow \quad \frac{\text{vol}(\overline{P_\sigma^l})}{2} = \text{vol}(p_\sigma^l) \leq M_{l,\varepsilon} \text{vol}(c)$$

where  $M_{l,\varepsilon}$  is as in Lemma 9.2.

*Proof.* For  $\sigma \in \mathcal{Z}$ , the set  $\overline{P_\sigma^l}$  is as described in (136), (137), (138) and then we can choose  $p_\sigma^l$  to be  $\exp(i[u_\sigma, v_\sigma])$  and  $\mathbb{S}^1$  can be divided as follows (for any  $j \in \mathbb{Z}$ ):

$$(195) \quad \begin{aligned} \mathbb{S}^1 = & e^{i\pi[\phi_\sigma(s_j[-1]), \phi_\sigma(s_{j+1}[-1])]} \cup e^{i[\pi\phi_\sigma(s_{j+1}[-1]), u_\sigma]} \cup p_\sigma^l \cup e^{i(v_\sigma, \pi\phi_\sigma(s_j))} \\ & \cup -e^{i\pi[\phi_\sigma(s_j[-1]), \phi_\sigma(s_{j+1}[-1])]} \cup -e^{i[\pi\phi_\sigma(s_{j+1}[-1]), u_\sigma]} \cup \left(-p_\sigma^l\right) \cup -e^{i(v_\sigma, \pi\phi_\sigma(s_j))} \end{aligned}$$

Furthermore, let  $\gamma$  be a closed  $\varepsilon$ -arc with  $P_\sigma^l \cap \gamma = \emptyset$ , then using (137) one easily sees that  $\gamma \subset \exp(i\pi(\phi_\sigma(s_j), \phi_\sigma(s_{j+1})))$  or  $-\gamma \subset \exp(i\pi(\phi_\sigma(s_j), \phi_\sigma(s_{j+1})))$  for some  $j \in \mathbb{Z}$  and therefore  $\pi(\phi_\sigma(s_{j+1}) - \phi_\sigma(s_j)) > \text{vol}(\gamma) = \pi\varepsilon$ , hence by Lemma 9.2 follow the inequalities (185) and

$$(196) \quad \gamma \subset e^{i\pi[\phi_\sigma(s_j[-1]), \phi_\sigma(s_{j+1}[-1])]} \quad \text{or} \quad -\gamma \subset e^{i\pi[\phi_\sigma(s_j[-1]), \phi_\sigma(s_{j+1}[-1])]}.$$

Therefore, taking into account the disjoint union (195) we see that the four components of  $\mathbb{S}^1 \setminus \{\gamma \cup -\gamma \cup p_\sigma^l \cup -p_\sigma^l\}$  can be ordered as  $c_1, c_2, -c_1, -c_2$  so that:  $c_1 \supset e^{i[\pi\phi_\sigma(s_{j+1}[-1]), u_\sigma]}$ ,  $c_2 \supset e^{i(v_\sigma, \pi\phi_\sigma(s_j))}$ , in particular:

$$(197) \quad \text{vol}(\pm c_1) \geq u_\sigma - \pi\phi_\sigma(s_{j+1}[-1]) \quad \text{vol}(\pm c_2) \geq \pi\phi_\sigma(s_j) - v_\sigma$$

and the corollary follows from (185).  $\square$

*Proof of Proposition 9.1* From Remark 4.15 and Subsection 7.6 we see that:

$$(198) \quad \|\mathcal{T}_{l_i}\|_\varepsilon = \left\| D^b(K(l_i)) \right\|_\varepsilon = K_\varepsilon(l_i)$$

hence the proposition follows for  $n = 1$ .

Assume that we have already proved the proposition for  $1 \leq n \leq N$ . And assume that  $\mathcal{T} = \mathcal{T}_{l_1} \oplus \mathcal{T}_{l_2} \oplus \cdots \oplus \mathcal{T}_{l_N} \oplus \mathcal{T}_{l_{N+1}}$ , where  $\mathcal{T}_{l_i} \cong D^b(K(l_i))$  and denote by  $L$  the set  $L = \{l_1, l_2, \dots, l_N, l_{N+1}\}$ .

If  $1 \leq l_j \leq 2$  for some  $j$ , then  $\|\mathcal{T}_{l_j}\|_\varepsilon = \|D^b(K(l_j))\|_\varepsilon = 0$ , and the statement follows from the induction assumption, Corollary 5.6, and  $\|\mathcal{T}_{l_j}\|_\varepsilon = 0$ . Therefore we can assume that all integers in  $L$  are at least 3. From the induction assumption there exists  $\delta > 0$  such that:

$$(199) \quad \begin{aligned} \delta + X &= \pi(1 - \varepsilon), \quad \text{where} \\ X &= \max \left\{ \|\mathcal{T}_{x_1} \oplus \mathcal{T}_{x_2} \oplus \cdots \oplus \mathcal{T}_{x_j}\|_\varepsilon : j < N + 1, x_i \in L \text{ for } 1 \leq i \leq j \right\} \end{aligned}$$

Note that due to Remark 4.2, Proposition 5.2 (d), and Corollary 5.5 for any sequence  $x_1, x_2, \dots, x_j$  in  $L$  holds:

$$(200) \quad \begin{aligned} & \|\mathcal{T}_{x_1} \oplus \mathcal{T}_{x_2} \oplus \cdots \oplus \mathcal{T}_{x_j}\|_\varepsilon = \\ & = \sup \left\{ \frac{\text{vol} \left( \bigcup_{i=1}^j \overline{P_{\sigma_i}^{x_i}} \right)}{2} : \exists \text{ closed } \varepsilon\text{-arc } \gamma \text{ s.t. } \forall i \sigma_i \in \text{Stab}(D^b(K(x_i))) \text{ and } \emptyset = P_{\sigma_i}^{x_i} \cap \gamma \right\} \end{aligned}$$

Assume now that  $\sigma_i \in \text{Stab}(D^b(K(l_i)))$  for  $i = 1, \dots, N + 1$  and that there exists a closed  $\varepsilon$ -arc  $\gamma$  satisfying  $\emptyset = P_{\sigma_i}^{l_i} \cap \gamma = \emptyset$  for  $i = 1, \dots, N + 1$ . In particular we can represent the circle  $\mathbb{S}^1$ :

$$(201) \quad \mathbb{S}^1 = \exp(i(\alpha, \beta)) \cup \gamma \cup -\exp(i(\alpha, \beta)) \cup -\gamma \quad \text{disjoint union}$$



where  $\alpha \in \mathbb{R}$  and  $\beta = \alpha + \pi(1 - \varepsilon)$ . If for some  $k$  the corresponding  $\sigma_k \notin \mathcal{Z}_{l_k} \subset \text{Stab}(D^b(K(l_k)))$ , then by Lemma 9.2  $\overline{P_{\sigma_k}^{l_k}}$  is finite and taking into account (199), (200) we derive:

$$(202) \quad \frac{\text{vol}\left(\bigcup_{i=1}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} = \frac{\text{vol}\left(\bigcup_{i=1, i \neq k}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} \leq X,$$

otherwise for all  $i$  we have  $\sigma_i \in \mathcal{Z}_{l_i}$ , and then by Corollary 9.3  $\frac{\text{vol}\left(\bigcup_{i=1}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} = \text{vol}\left(\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}\right)$ , where  $p_{\sigma_i}^{l_i}$  is a closed arc as explained in Corollary 9.3 and we can assume that  $p_{\sigma_i}^{l_i} \subset \exp(i(\alpha, \beta))$  for all  $i$  (see (201)). There exist  $\delta_- > 0$ ,  $\delta_+ > 0$  such that  $\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i} \subset \exp(i[\alpha + \delta_-, \beta - \delta_+])$ ,  $\exp(i(\alpha + \delta_-), \exp(i(\beta - \delta_+))) \in \bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}$ . Let  $j, k$  be such that  $\exp(i(\alpha + \delta_-)) \in p_{\sigma_j}^{l_j}$  and  $\exp(i(\beta - \delta_+)) \in p_{\sigma_k}^{l_k}$ . If we denote  $M = \max\{M_{l_i, \varepsilon} : 1 \leq i \leq N+1\}$ , then from Corollary 9.3 we obtain  $\text{vol}(p_{\sigma_j}^{l_j}) + \text{vol}(p_{\sigma_k}^{l_k}) \leq M(\delta_+ + \delta_-)$ . Since  $\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i} \subset \exp(i[\alpha + \delta_-, \beta - \delta_+])$  it follows that  $\delta_+ + \delta_- \leq \pi(1 - \varepsilon) - \text{vol}\left(\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}\right)$ , therefore we can write:

$$(203) \quad \begin{aligned} \frac{\text{vol}\left(\bigcup_{i=1}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} &= \text{vol}\left(\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}\right) \leq \text{vol}(p_{\sigma_j}^{l_j}) + \text{vol}(p_{\sigma_k}^{l_k}) + \text{vol}\left(\bigcup_{i=1, i \neq j, i \neq k}^{N+1} p_{\sigma_i}^{l_i}\right) \\ &\leq M \left( \pi(1 - \varepsilon) - \text{vol}\left(\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}\right) \right) + \frac{\text{vol}\left(\bigcup_{i=1, i \neq j, i \neq k}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} \\ &\leq M \left( \pi(1 - \varepsilon) - \text{vol}\left(\bigcup_{i=1}^{N+1} p_{\sigma_i}^{l_i}\right) \right) + X = M \left( \pi(1 - \varepsilon) - \frac{\text{vol}\left(\bigcup_{i=1}^{N+1} \overline{P_{\sigma_i}^{l_i}}\right)}{2} \right) + X \end{aligned}$$

The obtained inequalities (202), (203), and the formula (200) with  $x_i = l_i$ , for  $i = 1, 2, \dots, N+1$  show that for a certain set  $Y$  and a real function  $G$  on  $Y$  we have:

$$\begin{aligned} \|\mathcal{T}_{l_1} \oplus \mathcal{T}_{l_2} \oplus \dots \oplus \mathcal{T}_{l_{N+1}}\|_\varepsilon &= \sup\{G(y) : y \in Y\} \\ \forall y \in Y \quad 0 \leq G(y) \leq \pi(1 - \varepsilon); \quad G(y) &\leq M(\pi(1 - \varepsilon) - G(y)) + X. \end{aligned}$$

Now recalling (199) we get  $G(y) \leq M(\pi(1 - \varepsilon) - G(y)) + \pi(1 - \varepsilon) - \delta$  for any  $y \in Y$ , which is the same as  $G(y) \leq \pi(1 - \varepsilon) - \frac{\delta}{M+1}$ . Therefore the proof completes with the following inequality:

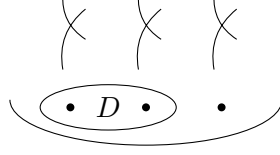
$$(204) \quad 0 < \|\mathcal{T}_{l_1} \oplus \mathcal{T}_{l_2} \oplus \dots \oplus \mathcal{T}_{l_{N+1}}\|_\varepsilon \leq \pi(1 - \varepsilon) - \frac{\delta}{M+1}.$$

## 10. A-SIDE INTERPRETATION AND HOLOMORPHIC SHEAVES OF CATEGORIES

In this section we give a different point of view on the category of representations of the Kronecker quiver and introduce the notion of holomorphic families of Kronecker quivers.

We suggest a framework in which sequences of holomorphic families of categories are viewed as sequences of extensions of non-commutative manifolds by relating our norm to the notion of holomorphic family of categories introduced by Kontsevich. Several questions and conjectures are posed.

First we sketch how to interpret  $D^b(K(n))$  as a perverse sheaf of categories. Recall that LG model of  $\mathbb{P}^2$  is  $\mathbb{C}^{*2}$ ,  $w = x + y + \frac{1}{xy}$  - see [1].



The category  $D^b(K(3))$  can be obtained by taking the part of the Landau Ginzburg model over a disc  $D$  which contains 2 singular fibers.

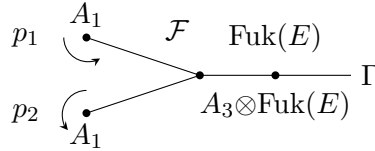
A surgery on the manifold:



results in changing the Floer homology  $\text{HF}(L_1, L_2) = 3$  to  $\text{HF}(L_1, L_2) = 4$ .

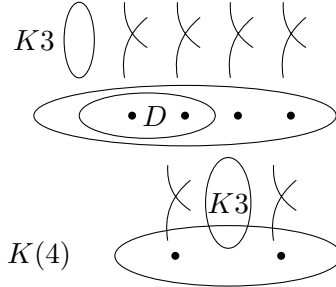


As a result we get  $D^b(K(4))$ . By similar surgeries we can get all quivers from  $K(0) = A_1 + A_1$  to  $K(n)$ . To interpret  $D^b(K(n))$  as a perverse sheaf of categories one considers a locally constant sheaf of categories over a graph  $\Gamma$  shown on the picture below, the picture encodes also the data about the sheaf, in particular  $p_1, p_2$  denote spherical functors (see [23], [1]):

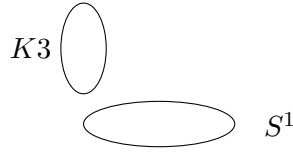


The category of global sections  $H^0(\Gamma, \mathcal{F})$  of the sheaf  $\mathcal{F}$  is the same as  $D^b(K(n))$ . The surgeries are recorded by the changes of the spherical functors  $p_1, p_2$ .

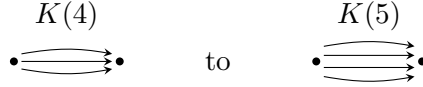
The category  $D^b(K(4))$  can be interpreted also as part of the LG model of  $\mathbb{P}^3$ ,  $\mathbb{C}^{*3}$ ,  $w = x + y + z + \frac{1}{xyz}$ :



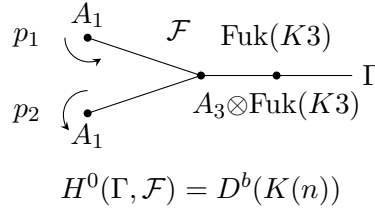
We make a surgery on the fiber - a  $K3$  surface:



This surgery amounts to change from



The Landau Ginzburg models with  $K3$  surfaces in the fibers can be interpreted as perverse sheafs of categories, encoded in the following picture - see [23]:



**Remark 10.1.** *The property of having a phase gap, which we require in this paper to define the norm, can also be interpret as existence of a CY form with certain properties.*

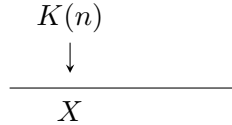
*Namely let  $Y$  be a LG model,  $\Omega$  is a CY form on  $Y$ . Let  $L$  be a Lagrangian s.t.  $\theta_1 \leq \arg \Omega|_L \leq \theta_2$ . Assume that there exists a form  $\beta$  on  $Y$  s.t.*

*(1)  $\beta = d\alpha$ , ( $\alpha$  is an  $n - 1$  form), (2)  $\text{Re} \beta|_L > 0$ . (3)  $\alpha \rightarrow 0$  when  $\omega \rightarrow 0$ .*

*Then there are no stable lagrangians  $L$  with  $\theta_1 \leq \arg \Omega|_L \leq \theta_2$ . In other words existence of such forms  $\Omega$  and  $\alpha$  lead to gaps in phases.*

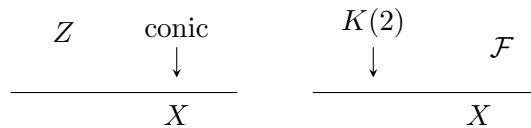
One more direction for future research is holomorphic families of categories, in particular holomorphic families of Kronecker quivers.

Holomorphic families of categories over  $X$  with fiber  $K(n)$  should be defined by homomorphisms  $\varphi_i : \mathcal{O}(U_i) \rightarrow \text{HH}^0(D^b(K(n)))$  in the zero-th Hochschild cohomology of  $D^b(K(n))$  where  $\{U_i\}$  is a covering of  $X$  by open sets. We use the following picture for such a holomorphic family of categories:



The holomorphic sheaves of categories are enhanced by perverse sheaves of stability conditions - see [24] for defining morphisms and the gluing between the categories on intersecting opens that defines the sheaf.

The case of holomorphic family of  $K(2)$  is the classical case of conic bundles:



The global sections  $H^0(X, \mathcal{F})$  are  $D^b(Z)$ . Similarly  $H^0(X, \mathcal{F})$  with  $K(n)$  for  $n \geq 3$  produces a new non-commutative variety.

Iterating the procedure described above results in a family of categories over a family of categories. Some questions addressing relations between the norms of the fibers and of the global sections follow:

**Question 10.2.** *Under what condition  $\|\mathcal{C}\| \geq \|H^0(X, \mathcal{F})\|$ ? (here  $\mathcal{C}$  is the category in the fiber)*

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{F} \\ \downarrow & & \\ \hline & X & \end{array}$$

**Question 10.3.** *Let us consider a tower of families of categories and each of the fiber categories  $\mathcal{C}_i$  has non maximal norm. Is it true that if the category in the combined fiber has a Rouquier dimension [2] equal to one then the norm of this category is nonmaximal?*

We summarise the proposed analogy in the table bellow.

Galois theory	Norms
$\begin{array}{c} X_2 \\ \downarrow \text{finite} \\ X_1 \\ \downarrow \text{finite} \\ X \end{array}$	$\begin{array}{c} X_2 \\ \downarrow \mathcal{C}_2 \quad \ \mathcal{C}_2\ _\epsilon < \max \quad \left( \begin{array}{c} \mathcal{C}_2 \\ \downarrow \\ \mathcal{C}_1 \end{array} \right) = \mathcal{C} \downarrow \\ X_1 \\ \downarrow \mathcal{C}_1 \quad \ \mathcal{C}_1\ _\epsilon < \max \\ X \end{array}$
<p>The sequence of finite coverings is finite</p>	<p>Rouquier dim <math>(\mathcal{C}) = 1</math> <math>\Downarrow</math> <math>\ \mathcal{C}\ _\epsilon &lt; \max</math></p>

**Question 10.4.** *Do we have a similar theory as the classical theory of conic bundles for sheaves of categories with fibers categories of representations of Kronecker quivers or any other quiver category with a Rouquier dimension [2] equal to one?*

In a certain way our norm can be seen as analogue of height function defined in [3]. We expect that some higher analogues of this norm for higher Rouquier dimensions can be defined. In fact in this paper we only scratch the surface proposing a possible approach to “noncommutative Galois theory” - representing “noncommutative manifolds” (categories) as a sequence of perverse sheaves of categories and holomorphic families of categories.

It will be interesting to study categories which can be represented as a tower of holomorphic families of categories with nonmaximal norms. One example of such category is  $D^b(\mathbb{P}^1 \times \dots \times \mathbb{P}^1)$ .

**Question 10.5.** *Characterise projective varieties  $X$  whose derived categories  $D^b(X)$  can be represented as tower of holomorphic families of categories with nonmaximal norms.*

1) Under what conditions are these projective varieties  $X$  rational? (It is rather clear that a nontrivial condition is needed since every hyperelliptic curve can be seen as such a tower. )

2) Can the existence of tower of holomorphic families of categories be represented in terms of modular forms?

In the end we put a question coming from the interplay between towers of sheaves of categories and stability conditions. Let  $K(N_1, N_2)$  be a category obtained from a family where the base is a category of representations of the Kronecker quiver  $K(N_1)$  and the fibers are the category of representations of the Kronecker quiver  $K(N_2)$ . Similarly we have  $K(N_1, N_2, \dots, N_l)$  denoting extensions of extensions.

**Question 10.6.** *Is the moduli space of stability condition of  $K(N_1, N_2, \dots, N_l)$  a bundle over Hilbert modular variety?*

It would be interesting to investigate the connection of the geometry of these Hilbert modular surfaces with the norms we have defined as well as new modular identities coming from wall-crossing formulae.

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